Another method to solve Beltrami's equation

Linear Beltrami Solver (LBS)

Let $M = (V, E, F)$ be simply-connected domain w/ boundary.

Let $V = \{(g_1, h_1), (g_2, h_2), \dots, (g_{|V|}, h_{|V|})\}$.

In discrete formulation, given $\mu = \rho + i\tau$, we want to compute a resulting mesh M' such that

$$v_n = (g_n, h_n) \mapsto w_n = (s_n, t_n)$$

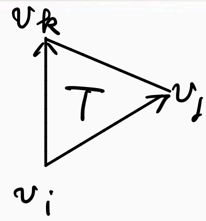
vertices in M'

On each face T, the discrete QC map f is linear.

$$\therefore \quad \underset{u+iv}{f|_T (x,y)} = \begin{pmatrix} u|_T(x,y) \\ v|_T(x,y) \end{pmatrix} = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{pmatrix}$$

$$\therefore \quad u_x|_T = a_T \,;\quad u_y|_T = b_T \,;\quad v_x|_T = c_T \,;\quad v_y|_T = d_T$$

Consider the directional derivatives along $v_j - v_i$ and $v_k - v_i$, we get:



$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{pmatrix}$$

Assume f is orientation-preserving, then:

$$\det \begin{pmatrix} g_j - g_i & g_k - g_i \\ h_j - h_i & h_k - h_i \end{pmatrix} = 2 \, \text{Area}(T).$$

$$\therefore \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \frac{1}{2\,\text{Area}(T)} \begin{pmatrix} s_j - s_i & s_k - s_i \\ t_k - t_i & t_k - t_i \end{pmatrix} \begin{pmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{pmatrix}$$

$$\begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} = \begin{pmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{pmatrix}$$

$$\begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix} = \frac{1}{2 \cdot Area(T)} \begin{bmatrix} s_j - s_i & s_k - s_i \\ t_j - t_i & t_k - t_i \end{bmatrix} \begin{bmatrix} h_k - h_i & g_i - g_k \\ h_i - h_j & g_j - g_i \end{bmatrix}$$
$$= \begin{bmatrix} A_T^i s_i + A_T^j s_j + A_T^k s_k & B_T^i s_i + B_T^j s_j + B_T^k s_k \\ A_T^i t_i + A_T^j t_j + A_T^k t_k & B_T^i t_i + B_T^j t_j + B_T^k t_k \end{bmatrix}.$$

where

$$A_T^i = (h_j - h_k)/2 \cdot Area(T); \quad A_T^j = (h_k - h_i)/2 \cdot Area(T); \quad A_T^k = (h_i - h_j)/2 \cdot Area(T);$$
$$B_T^i = (g_k - g_j)/2 \cdot Area(T); \quad B_T^j = (g_i - g_k)/2 \cdot Area(T); \quad B_T^k = (g_j - g_i)/2 \cdot Area(T).$$

Now, define : $\mathrm{Div}(\underbrace{X_1, X_2}_{\vec{V}=(X_1,X_2)})(v_i) = \sum_{T \in N_i} Area(T) \cdot A_T^i X_1(T) + Area(T) \cdot B_T^i X_2(T)$

$\vec{V} = (X_1, X_2)$ vector field defined on each face T

All faces around v_i

Easy to check: $\mathrm{Div}(-d, c) = \sum_{T \in N_i} -Area(T)\, A_T^i \left(B_T^i t_i + B_T^j t_j + B_T^k t_k\right) + Area(T)\, B_T^i \left(A_T^i t_i + A_T^j t_j + A_T^k t_k\right)$

$-d = -v_y$, $c = v_x$

$= 0$

Similarly, $\mathrm{Div}(-b, a) = 0$

Recall that:

$$\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \underbrace{\frac{1}{1-\rho^2-\tau^2}\begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix}\begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix}}_{A}\begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = A \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$\therefore$ $\mathrm{Div}\left(A\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right) = 0$ to solve for u with suitable boundary conditions $\left(\Leftrightarrow \mathrm{Div}\left\{ A \begin{bmatrix} B_T^i s_i + B_T^j s_j + B_T^k s_k \\ B_T^i t_i + B_T^j t_j + B_T^k t_k \end{bmatrix} \right\} = 0 \right)$

(If $M = [0,1] \times [0,1]$, we set $u = 0$ on the left boundary
and $M' = [0,1] \times [0,h]$ for some h. $u = 1$ on the right boundary)

Once u is determined, we can determine

$$h = \sum_T \left(\alpha_T (u_x)_T^2 + 2\beta_T (u_x)_T (u_y)_T + \gamma_T (u_y)_T^2 \right)$$

v can be determined by solving:

$$\mathrm{Div}\left(A \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right) = 0$$

with boundary conditions: $v = 0$ on the bottom boundary

$v = h$ on the upper boundary.

Remark: In case landmark constraints are imposed, we can solve:

$$\mathrm{Div}\left(A\begin{pmatrix}u_x\\u_y\end{pmatrix}\right)=0 \quad \text{and} \quad \mathrm{Div}\left(A\begin{pmatrix}v_x\\v_y\end{pmatrix}\right)=0$$

subject to $u(p_i)=q_i^u$ and $v(p_i)=q_i^v$ for $i=1,2,\ldots,m$ (by substituting them into the linear system)

where $\{p_i\}_{i=1}^m \leftrightarrow \{q_i = q_i^u + i q_i^v\}_{i=1}^m$ denotes the landmark corresponding. It gives a q.c. map whose BC is close to μ. (not exactly same)

$$B\begin{pmatrix}u_1\\u_2\\ \vdots \\ u_j \leftarrow q_j^u \\ \vdots \leftarrow q_\ell^u \\ u_N\end{pmatrix}=0 \quad \text{and} \quad B'\begin{pmatrix}v_1\\v_2\\ \vdots \\ v_N\end{pmatrix}=0$$

etc

(u_i = value of u at vertex i etc)

$\Rightarrow B_1\vec{u}=\vec{b}_1$ etc ...

Fixing conformality distortion for Fast Spherical Conformal Parameterization

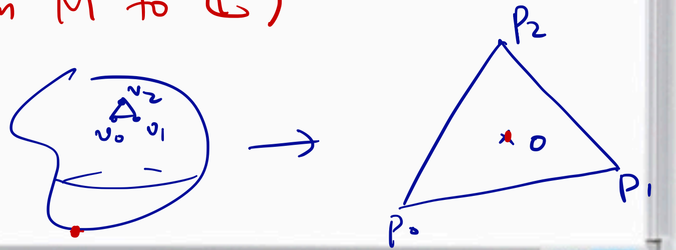
Recall: Given genus-0 mesh $M=(V,E,F)$, we can take away one small triangle $\Delta = [v_0, v_1, v_2]$ (treat it as north pole) and map it to big triangle $T=[p_0,p_1,p_2]$ (w/ same angle structure as Δ) by solving:

$$\sum_{[v_i,v_j]\in E} w_{ij}\left(f(v_j)-f(v_i)\right)=0$$

subject to the constraint that $f(v_0)=p_0\in\mathbb{C}$, $f(v_1)=p_1\in\mathbb{C}$ and $f(v_2)=p_2\in\mathbb{C}$.

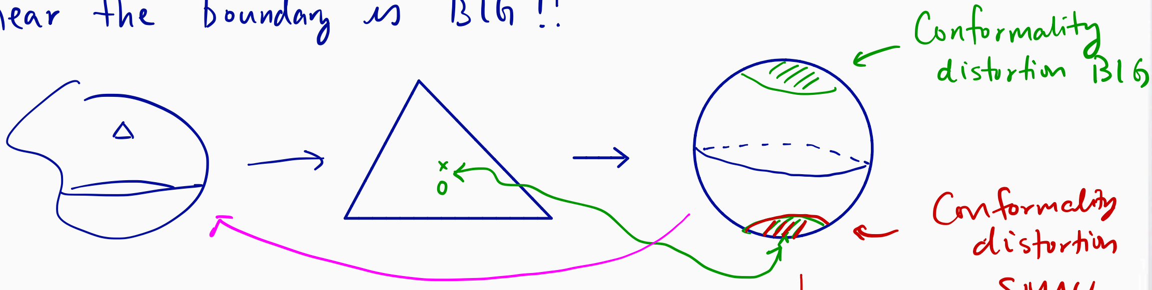
(f is a piecewise linear map from M to $\mathbb{C}$)

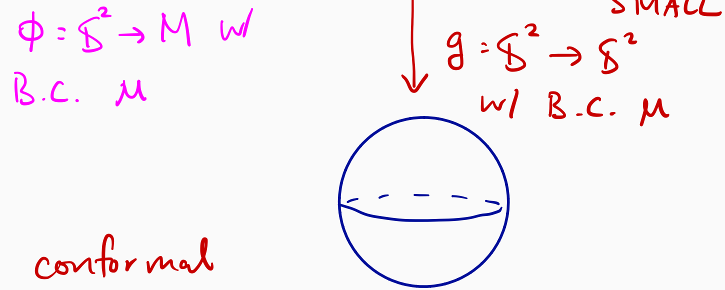
(Linear system = fast)



Drawback: Conformality near the origin is small but conformality near the boundary is BIG!!

Strategy:

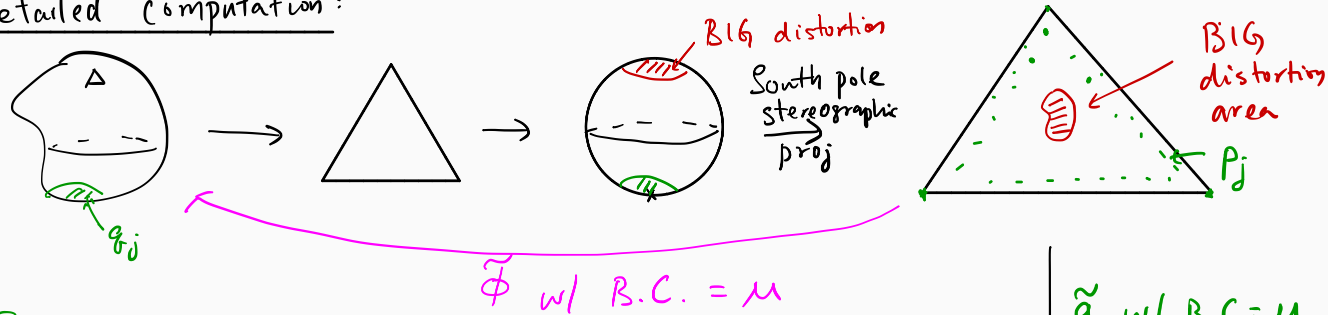


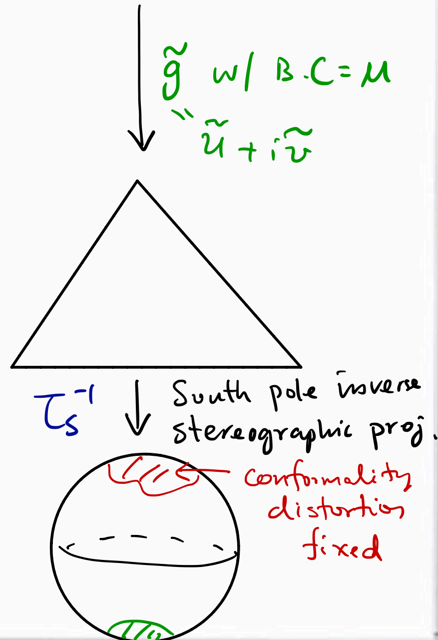


Then: $\phi \circ g^{-1} : S^2 \to M$ has B.C. $= 0$ and hence conformal!!

Computing $g : S^2 \to S^2$ involves conformal chart. Use sterographic projection!!

Detailed computation:





Solve:

$$\mathrm{Div}\left(A\begin{pmatrix}\tilde u_x\\ \tilde u_y\end{pmatrix}\right)=0 \quad \text{and} \quad \mathrm{Div}\left(A\begin{pmatrix}\tilde v_x\\ \tilde v_y\end{pmatrix}\right)=0$$

Subject to $\tilde g(p_j) = q_j$ for $j=1,2,$

Then: $\tilde\Phi \circ \tilde g^{-1} \circ T_s$ has less conformality distortion near north pole!