Fast algorithm for genus-0 spherical conformal parameterization

Idea: Let S be a Riemann surface. $\exists$ conformal parameterization $\ni$ $\phi : S \to \mathbb{S}^2$.

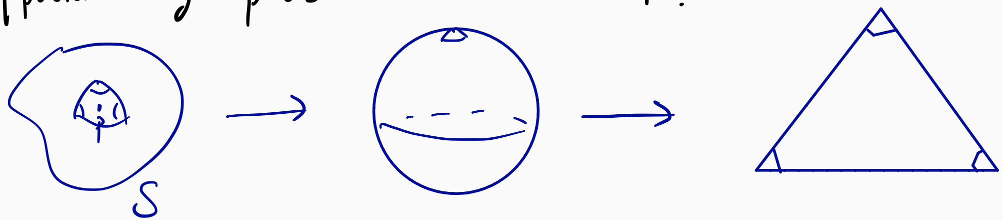
Let $p \in S$. We can assume $\phi(p) =$ north pole.

Let τ be the sterographic projection.

Let Δ be a small "curved" triangle around p $\ni$

$$\underbrace{\tau \circ \phi}_{\tilde{\phi}} (\Delta) = \tilde{\Delta} = \text{big triangle in } \mathbb{C}.$$

The angles at the 3 vertices of Δ is approximately preserved under $\tilde{\phi}$.



Method: In the discrete case, let $M = (V, E, F)$. Take $T = [v_0, v_1, v_2] \in F$ and let $p \in T$ be the centroid of T.

We can find a harmonic map with boundary condition that $\tilde{\Phi}(v_0) = w_0$, $\tilde{\Phi}(v_1) = w_1$ and $\tilde{\Phi}(v_2) = w_2 \ni$ $[w_0, w_1, w_2]$ has the same angle structure as $[v_0, v_1, v_2]$.

Mathematically, we need to solve:

$$\sum_{[v_i, v_j] \in E} w_{ij} \left(f(v_j) - f(v_i) \right) = 0 \quad \text{for } \forall\, i = 1, 2, \ldots, N$$

and fix $f(v_0) = w_0$, $f(v_1) = w_1$ and $f(v_2) = w_2$.

(Linear system, much faster than iterative scheme)

Remark: Numerical error (conformality distortion) near the north pole is big.

We'll use quasiconformal theories to fix it.

Idea: Let $\varphi: S \to \mathbb{S}^2$ with big distortion at north pole.

Reparameterize φ by quasi-conformal map $g \ni$

$g \circ \varphi$ can fix the conformality distortion

Brain landmark matching optimized harmonic parameterization

Goal: Given a brain cortical surface S. Let $\{p_i\}_{i=1}^N$ be landmark points defined on S. Want to find: $f: S \to \mathbb{S}^2$ such that f is as conformal / harmonic as possible and $f(p_i) = q_i$ $(i=1,2,\dots,m)$ for some fixed locations $q_i \in \mathbb{S}^2$.

Suppose S_1 and S_2 be two brain surfaces w/ landmarks $\{p_i\}_{i=1}^m$ and $\{p_i'\}_{i=1}^m$ respectively. Let $f: S_1 \to \mathbb{S}^2$ and $f': S_2 \to \mathbb{S}^2$ $\ni$ $f(p_i) = q_i = f'(p_i')$ for $i=1,2,\dots,m$.

Then, $(f')^{-1} \circ f : S_1 \to S_2$ is a landmark-matching surface registration of S_1 and S_2 (Atlas-based surface registration)

Method 1: Find $f \ni \sum_{[v_i,v_j]\in E} w_{ij}\left(f(v_j)-f(v_i)\right) \quad \forall i=1,2,\dots,n$

and $f(p_i)=q_i \quad i=1,2,\dots,m$

(if p_i's are vertices)

Drawback: Bijectivity is difficult to control.

Method 2: Find f that minimizes:

$$E_{landmark}(f) = \frac{1}{2}\sum_{[v_i,v_j]\in E} w_{ij}\left|f(v_i)-f(v_j)\right|^2 + \lambda\sum_{k=1}^{m}\left|f(p_k)-q_k\right|^2$$

λ = adjusting parameter (Big if we want more accurate landmark matching)

Soft constraint can better control bijectivity.

Using same idea, we use descent method to minimize $E_{landmark}$

$$\frac{d\vec{f}}{dt} = -\mathcal{D}\vec{f}, \quad \text{where}$$

$$(\widetilde{\mathcal{D}\vec{f}})_i = \sum_{[v_i,v_j]\in} w_{ij}\,(f(v_j) - f(v_i)) + 2\lambda \sum_{k=1}^{m} (f(p_k) - q_k)$$

Normalize $(\widetilde{\mathcal{D}\vec{f}})$ to its tangential component to get

$$\mathcal{D}\vec{f} = (\widetilde{\mathcal{D}\vec{f}}) - \langle \widetilde{\mathcal{D}\vec{f}}, \vec{n} \rangle \vec{n}$$

Iteratively adjust $\vec{f}$ to minimize $E_{landmark}$.

Remark: Both methods do not have bijectivity guarantee.

Use quasiconformal theories to fix it.

Quasiconformal map between Riemann surfaces

Basic idea: Given two Riemann surfaces S_1 and S_2.
Under the conformal coordinate charts, $f: S_1 \to S_2$ is "quasi-conformal" iff f is "quasi-conformal" as a map from $\mathbb{C} \to \mathbb{C}$ under the conformal charts (follows from the definition. Later)

Suppose S_1 and S_2 are simply-connected open surfaces.
$\exists$ conformal $\phi_1 : \mathbb{D} \to S_1$ and $\phi_2 : \mathbb{D} \to S_2$ (Global Conformal parameterization)

Then: $f: S_1 \to S_2$ is quasiconformal iff

$\phi_2^{-1} \circ f \circ \phi_1 : \mathbb{D} \to \mathbb{D}$ is quasi-conformal in 2D.

$\therefore$ Focus our attention on $\mathbb{C} \to \mathbb{C}$ first!

Quasi-conformal map from $\mathbb{C}$ to $\mathbb{C}$

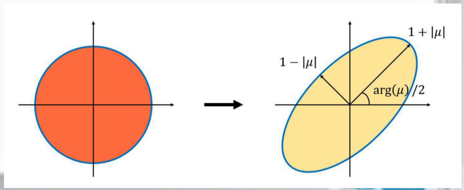
Definition: (Quasiconformal map) Let $f: \mathbb{C} \to \mathbb{C}$ be a C^1 homeomorphism. f is called a quasi-conformal map with respect to a complex-valued function $\mu: \mathbb{C} \to \mathbb{C}$, called the Beltrami coefficient, with $\|\mu\|_\infty < 1$ & :

$$(*) \quad \frac{\partial f}{\partial \bar{z}}(z) = \mu(z) \frac{\partial f}{\partial z} \quad \text{where}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$$

$\mu(z)$ measures the local geometric distortion at z.

$(*)$ is called the Beltrami's equation



Remark: 1. When $\mu \equiv 0$, the Beltrami's equation is reduced to the Cauchy-Riemann equation. Let $f = u + iv$ (u, v real functions)

Then: $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - i\frac{\partial}{\partial y}(u+iv)\right)$$
$$= \frac{1}{2}\left((u_x + v_y) + i(v_x - u_y)\right) = 0$$

$$\Rightarrow \begin{cases} u_x = -v_y \\ u_y = +v_x \end{cases} \quad \text{(Cauchy-Riemann eqt)}$$

2. In matrix form, a conformal/holomorphic complex-value function $f = u + iv$ satisfies:

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

Or $\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \overset{Id}{\begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$. —— $(**)$

Quasi-conformal map generalizes $(**)$ by considering

$$\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}}_{\text{Represent the metric distortion}} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

for some α, β and γ depending on μ.

3. Let $J(z)$ = Jacobian of $f = u + iv$ at z.

Then $J = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$

Note that:

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \frac{(u_x + v_y)^2 + (v_x - u_y)^2}{4} - \frac{(u_x - v_y)^2 + (v_x + u_y)^2}{4}$$

$$= (u_x v_y - u_y v_x) = J(z)$$

$$\therefore J(z) = \left|\frac{\partial f}{\partial z}\right|^2 \left(1 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \Big/ \left|\frac{\partial f}{\partial z}\right|^2\right) = \left|\frac{\partial f}{\partial z}\right|^2 \left(1 - |\mu(z)|^2\right)$$

Thus, if $\|\mu(z)\|_\infty < 1$ and $\left|\frac{\partial f}{\partial z}\right| \neq 0$ (f = homeomorphism) then $J(z) > 0$ everywhere. $\therefore$ f is orientation-preserving everywhere

Existence and Uniqueness Theorem

Theorem: (Measurable Riemann mapping theorem) Suppose $\mu : \mathbb{C} \to \mathbb{C}$ is Lebesgue measurable and satisfies $\|\mu\|_\infty < 1$, then there exists a quasi-conformal homeomorphism ϕ from $\mathbb{C}$ onto itself, which is in the Sobolev space $W^{1,2}(\mathbb{C})$ and satisfies the Beltrami equation $\left(\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}\right)$ in the distribution sense. Also, by fixing $0, 1, \infty$, the associated quasiconformal homeomorphism ϕ is uniquely determined.

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Theorem: Suppose $\mu : \mathbb{D} \to \mathbb{C}$ is Lebesgue measurable and satisfies $\|\mu\|_\infty < 1$. Then, there exists a quasiconformal homeomorphism ϕ from $\mathbb{D}$ to itself, which is in the Sobolev space $W^{1,2}(\Omega)$ and satisfies the Beltrami equation in the distribution sense. Also, by fixing 0 and 1, ϕ is uniquely determined.

Proof: Follows from previous thm by reflection.
(Based on Beltrami holomorphic flow Later!)

Composition of quasiconformal maps

Let $f: \mathbb{C} \to \mathbb{C}$ and $g: \mathbb{C} \to \mathbb{C}$ be quasiconformal maps. Then, the Beltrami coefficient of the composition map $g \circ f$ is given by:

$$\mu_{g\circ f}(z) = \frac{\mu_f(z) + \overline{f_z(z)}/f_z(z)\,(\mu_g \circ f)}{1 + \overline{f_z(z)}/f_z(z)\,\overline{\mu_f}\,(\mu_g \circ f)}.$$

Theorem: Let $f: \Omega_1 \to \Omega_2$ and $g: \Omega_2 \to \Omega_3$ be quasiconformal maps. Suppose the Beltrami coefficients of f^{-1} and g are the same. Then the Beltrami coefficient of $g \circ f$ is equal to 0 and $g \circ f$ is conformal.

Proof: Note that: $\mu_{f^{-1}} \circ f = -\left(\overline{f_z}/|f_z|\right)^2 \mu_f$.

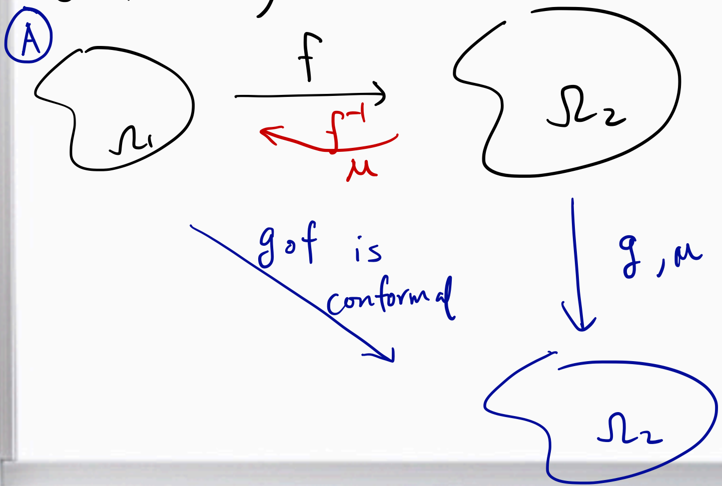
$\because$ $\mu_{f^{-1}} = \mu_g$, we have:

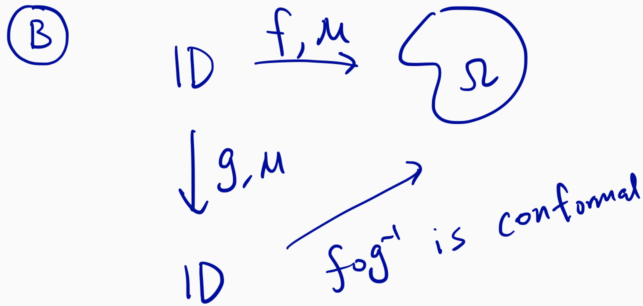
$$\mu_f + \left(\overline{f_z}/f_z\right)(\mu_g \circ f) = \mu_f + \left(\overline{f_z}/f_z\right)(\mu_{f^{-1}} \circ f)$$

$$= \mu_f + \left(\overline{f_z}/f_z\right)\left(-\frac{f_z}{\overline{f_z}}\right)\mu_f = 0$$

By the composition formula, $\mu_{g\circ f} = 0$ and so $g\circ f$ is conformal.

<u>Remark</u>: The above theorem gives a useful way to fix conformality distortion.





In depth analysis of Beltrami's equation

Let $f = u + iv$ and $\mu = \rho + i\tau$. Comparing the real and imaginary parts of $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ gives:

$$\begin{pmatrix} \rho - 1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \rho+1 & \tau \\ \tau & 1-\rho \end{pmatrix} \begin{pmatrix} -v_y \\ v_x \end{pmatrix}.$$

$\because \|\mu\|_\infty < 1$, $\det \begin{pmatrix} \rho+1 & \tau \\ \tau & 1-\rho \end{pmatrix} = 1 - \rho^2 - \tau^2 > 0$ for $\forall z \in \Omega$.

$$\therefore \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \frac{1}{1-\rho^2-\tau^2} \begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix} \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Denote $C = \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix}$. We get $\begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \frac{1}{1-\rho^2-\tau^2} C^T C \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

where

$$-A = \frac{-1}{1-\rho^2-\tau^2} \begin{pmatrix} 1-\rho & -\tau \\ -\tau & \rho+1 \end{pmatrix} \begin{pmatrix} \rho-1 & \tau \\ \tau & -(\rho+1) \end{pmatrix} = \frac{-1}{1-\rho^2-\tau^2} \begin{pmatrix} -(1-\rho)^2-\tau^2 & 2\tau \\ 2\tau & -\tau^2-(\rho+1)^2 \end{pmatrix}$$

Area distortion under quasi-conformal map

To simplify our discussion, let $f: \underbrace{[0,1]\times[0,1]}_{R} \to \Omega \subseteq \mathbb{C}$.

($\therefore$ Area of source domain R is 1)

Now, area of $\Omega = \int_R J(z)\, dz$

$$= \int_R (u_x v_y - v_x u_y)\, dz$$

Recall that $\begin{pmatrix} +v_y \\ -v_x \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \alpha u_x + \beta u_y \\ \beta u_x + \gamma u_y \end{pmatrix}$

$\therefore$ Area of $\Omega = \int_R u_x(\alpha u_x + \beta u_y) + (\beta u_x + \gamma u_y) u_y$

$$= \int_R \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2$$

where α, β and γ are determined by $\mu = \rho + i\tau$.

Remark: • μ (or α, β, γ) introduces area distortion under f

• Computationally, once u associated to μ is obtained, we can determine the area of the target domain by

$$A = \int_R \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2$$

If $\Omega = [0,1] \times [0,h]$, then $h = A$.

$\therefore$ Once u is computed, the geometry of the target domain can be determined.

$\therefore$ v can be computed (Useful observation!)