Lecture 3:

Recap

Gauss-Bonnet Theorem

Theorem: (Gauss-Bonnet) Let M be a compact closed surface.

$$\int_M K \, dA = 2\pi \underbrace{\chi(M)}_{\text{Euler characteristic}}$$

(integer depending on the topology)

Discrete Gauss-Bonnet Theorem

Theorem: For an oriented discrete triangulated surface M,

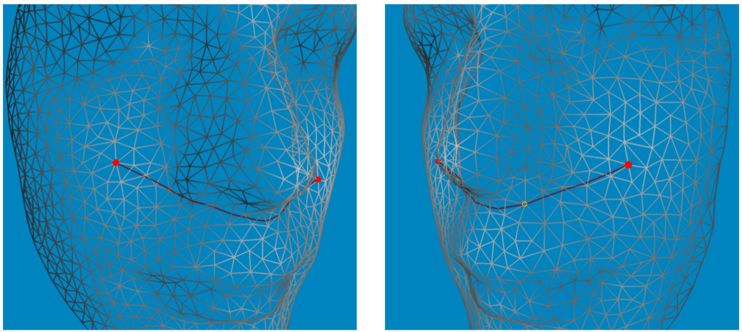
$$\sum_{v_i} K(v_i) = 2\pi\,\chi(M)$$

where $\{v_i\}$ is the collection of vertices, $K(\vec{v}_i)$ is the discrete Gaussian curvature defined as:

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases}$$

and

$$\chi(M) = \underbrace{|V|}_{\text{\# of vertices}} + \overbrace{|F|}^{\text{\# of faces}} - \underbrace{|E|}_{\text{\# of edges}}$$



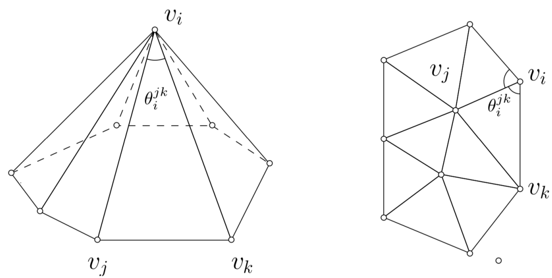


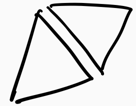
Figure: Discrete Gaussian curvature.

Proof: Let $M = (V, E, F)$. If M is closed, then:

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jk}$$

$$= 2\pi |V| - \pi |F|$$

$\because$ M is closed $\therefore$ $3|F| = 2|E|$



$$\therefore \chi(M) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F|$$

$$= |V| - \frac{1}{2}|F|$$

$$\therefore \sum_{v_i \in V} K(v_i) = 2\pi \, \chi(M).$$

Assume M has a boundary ∂M.

Let V_0 = interior vertex set

V_1 = boundary set

$\Big\}\ |V| = |V_0| + |V_1|$

E_0 = interior edge set

E_1 = boundary edge set

$\Big\}\ |E| = |E_0| + |E_1|$

$\because$ All boundary are closed loop $\therefore$ $|E_1| = |V_1|$.

Each interior edge is adjacent to two faces and each boundary edge is adjacent to one face, we have:

$$3|F| = 2|E_0| + |E_1| = 2|E_0| + |V_1|$$

$$\therefore \chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - \overset{|V_1|}{|E_1|}$$

$$= |V_0| + |F| - |E_0|$$

$\because$ $|E_0| = \frac{1}{2}(3|F| - |V_1|)$

$$\therefore \chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|.$$

$$\therefore \sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) = \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk} \right)$$

$$= 2\pi |V_0| + \pi |V_1| - \pi |F|$$

$$= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right)$$

$$= 2\pi \chi(M)$$

//

Basic theories of compact Riemann surface

Definition: (Harmonic function) Suppose $u: D \to \mathbb{R}$ is a real valued function defined on $D \subseteq \mathbb{C}$. If $u \in C^2(D)$ and for any $z \in D$, $z = x+iy$, we have:

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \quad \text{for } \forall z.$$

Then: u is a harmonic function.

Definition: (Holomorphic function) A function $f: \mathbb{C} \to \mathbb{C}$, $(x,y) \mapsto (u,v)$ is holomorphic if:

$$\begin{cases} \dfrac{\partial u}{\partial x}(z) = \dfrac{\partial v}{\partial y}(z) \\[2ex] \dfrac{\partial u}{\partial y}(z) = -\dfrac{\partial v}{\partial x}(z) \end{cases} \quad \text{for } \forall z \in \mathbb{C}$$

(Cauchy-Riemann eqt)

Remark: • Denote $dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Then: $\Delta = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ (Check!)

Also, f is holomorphic if $\dfrac{\partial f}{\partial \bar{z}} = 0$. (Check!)

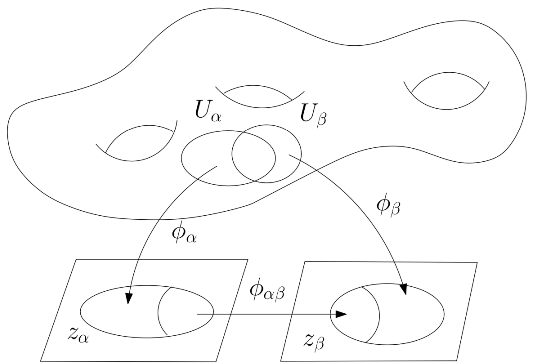
• If a holomorphic function is bijective and f^{-1} is also holomorphic, then f is called biholomorphic or conformal.

Definition: (Riemann surface) A Riemann surface S is a 2-dim manifold M with an atlas $\{(\mathcal{U}_\alpha, z_\alpha)\}$, such that $\{\mathcal{U}_\alpha\}$ is an open covering, $M \subset \bigcup_\alpha \mathcal{U}_\alpha$ and $z_\alpha : \mathcal{U}_\alpha \to \mathbb{C}$ is a homeomorphism from $\mathcal{U}_\alpha$ to an open set in $\mathbb{C}$, $z_\alpha(\mathcal{U}_\alpha)$. Also, if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$, then:

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to z_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is biholomorphic / conformal.

$\{(\mathcal{U}_\alpha, z_\alpha)\}$ is called the conformal atlas of S.



<u>Remark:</u> • Given two conformal atlas $\{(U_\alpha, z_\alpha)\}$ and $\{(V_\beta, \tau_\beta)\}$, if their union is also a conformal atlas, then we say $\{(U_\alpha, z_\alpha)\}$ is equivalent to $\{(V_\beta, \tau_\beta)\}$.

Each equivalence class of conformal atlas is called a <u>conformal structure.</u>

• Given a smooth manifold M, we can equip M with a Riemannian metric $g = (g_{ij})$, which gives the inner product in the tangent space $T_p(M)$,

$$g_{ij} = \langle \partial_i, \partial_j \rangle_g .$$

Its inverse matrix is (g^{ij}), satisfies $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$

$$= \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

• Suppose M has a Riemannian metric g. Then we require that on each chart of $\{(U_\alpha, z_\alpha)\}$:

$$g = e^{2\lambda(z_\alpha)} dz_\alpha d\overline{z_\alpha} = e^{2\lambda(z_\alpha)}(dx_\alpha^2 + dy_\alpha^2)$$

Recall: given $\vec{v} = v_1 \frac{\partial}{\partial x_\alpha} + v_2 \frac{\partial}{\partial y_\alpha} \in T_pM$

$\vec{w} = w_1 \frac{\partial}{\partial x_\alpha} + w_2 \frac{\partial}{\partial y_\alpha} \in T_pM$

Then: $(dx_\alpha^2 + dy_\alpha^2)(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2$

In this case, we say the local parameters associated to $\{(U_\alpha, z_\alpha)\}$ are isothermal coordinates.

Proposition: Given a metric surface with a differential atlas $\{(U_\alpha, z_\alpha)\}$. If all local coordinates are isothermal coordinates, then $\{(U_\alpha, z_\alpha)\}$ is a conformal structure.

Remark: Any metric surface has an isothermal coordinates.

Theorem: Any metric surface is a Riemann surface.

Definition: (Conformal mapping) Suppose M and $\tilde{M}$ are two Riemann surfaces. A homeomorphism $f: M \to \tilde{M}$ is called a conformal mapping, if $\forall p \in M$, $\tilde{p} = f(p) \in \tilde{M}$, for any local parameter chart (U, ϕ) and $(\tilde{U}, \tilde{\phi})$, $z = \phi(p)$, $\tilde{z} = \tilde{\phi}(\tilde{z})$,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"] & \tilde{M} \arrow[d, "\tilde{\phi}"] \\ z \arrow[r, "\tilde{\phi}\circ f\circ\phi^{-1}"] & \tilde{z} \end{tikzcd}$$

under local parameters

$$\tilde{z} = \tilde{\phi} \circ f \circ \phi^{-1}$$ is holomorphic in U.

Remark: Our goal is to compute conformal map from complicated surface M (Brain surface) to D (such as sphere, 2D rectangles, etc)

Remark: • If $\exists\, f : M \to \tilde{M}$, then M and $\tilde{M}$ are called conformally equivalent.

• Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $w = f(z)$.

Then: $$dw = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

and also $$\underbrace{dw\, d\bar{w}}_{\text{metric on the new "transformed" chart.}} = \left|\frac{\partial f}{\partial z}\right|^2 \underbrace{dz\, d\bar{z}}_{\text{metric on original chart}}$$

$= d\rho^2 + d\tau^2$ (if $w = \rho + i\tau$)

$\therefore$ "Transformed metric" under conformal map is the same as the original chart up to a scalar multiplication. $\left|\frac{\partial f}{\partial z}\right|^2$ is called the conformal factor.