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# Math 3093 Tutorial 11.

1. Define

$$u(x,t) := \frac{x}{t} H_t(x),$$

where $\mathcal{H}_t(x)$ is the heat kernel given by

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}},$$

Show that

(a) $u$ satisfies the heat equation for $t>0$.

(b) $\lim_{t\to 0^+} u(x,t) = 0$ for every $x \in \mathbb{R}$

(c). $u$ is not continuous at the origin

Sol$^{n}$:

1 (a).

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\frac{\partial}{\partial t}H_t(x)$$

$$= -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\left(\frac{1}{\sqrt{4\pi t}}\frac{1}{t}\cdot\left(-\frac{1}{2}\right)e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\left(\frac{x^2}{4t^2}\right)\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{t^2} - \frac{x}{2t^2} + \frac{x^3}{4t^3}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\mathcal{H}_t'(x)$$

$$= \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\mathcal{H}_t(x)\cdot\left(-\frac{x}{2t}\right)$$

$$= \mathcal{H}_t(x)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \mathcal{H}_t'(x)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t}\right)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t^2} + \frac{x^3}{4t^3} - \frac{x}{t^2}\right) = \frac{\partial u}{\partial t}$$

1(b). Fix $x \in \mathbb{R}$,

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{x}{t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

$$= \lim_{y\to\infty} xy \frac{\sqrt{y}}{\sqrt{4\pi}} e^{-\frac{x^2}{4}y} \qquad \left(y = \frac{1}{t}\right)$$

$$= \lim_{y\to\infty} \frac{x}{\sqrt{4\pi}} \frac{y^{3/2}}{e^{\frac{x^2}{4}y}}$$

$$= \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \text{ (L'Hôpital's rule)} \end{cases}$$

(c). For every $x \in \mathbb{R}$, $\lim_{t\to 0^+} u(x,t) = 0$, hence we may put $u(x,0) = 0 \quad \forall x \in \mathbb{R}$.

$u$ is not continuous at the origin:

$$\lim_{(x,t)\to(0,0)} u(x,t) = \lim_{(x,t)\to(0,0)} \frac{x}{t} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

We may consider a special path to the origin. For each $t>0$, put $x = \sqrt{t}$ so that $e^{-\frac{x^2}{4t}} = e^{-\frac{1}{4}}$ is a constant.

Now, $$\lim_{(\sqrt{t},t)\to(0,0)} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}} = \infty$$

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Q2.

Consider the following variant of the heat equation:

$$\begin{cases} x^2 \dfrac{\partial^2 u}{\partial x^2} + ax\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t}, & 0<x<\infty, \quad t>0 \\ u(x,0) = f(x), & \end{cases} \tag{1}$$

[ Hint: Make the change of variables $x = e^{-y}$ $\quad -\infty<y<\infty$
Set $v(y,t) = u(e^{-y},t)$ and $F(y) = f(e^{-y})$ ]

$$\frac{\partial v}{\partial y} = \left.\frac{\partial u}{\partial x}\right|_{(e^{-y},t)} \cdot (-e^{-y})$$

$$\frac{\partial^2 v}{\partial y^2} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{(e^{-y},t)} (-e^{-y})^2 + \left.\frac{\partial u}{\partial x}\right|_{(e^{-y},t)} (e^{-y})$$

$$\therefore \quad \left.\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x}(-x) \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 u}{\partial x^2}(x)^2 + \frac{\partial u}{\partial x}(x) \end{aligned}\right\}$$

$x = e^{-y}$

$\frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}$ evaluated at $(y,t)$ while

$\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ evaluated at $(x,t) = (e^{-y},t)$

$$\Rightarrow \begin{cases} x^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial v}{\partial y} \\ ax\dfrac{\partial u}{\partial x} = -a\dfrac{\partial v}{\partial y} \end{cases}$$

So, (1) becomes

$$(2) \quad \begin{cases} \dfrac{\partial^2 v}{\partial y^2} + (1-a)\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial t}, & -\infty<y<\infty, \quad t>0 \\ v(y,0) = F(y) & \end{cases}$$

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Taking Fourier transform in (2):

$$\frac{\partial v}{\partial y} \longmapsto (2\pi i \xi)\, \hat{v}(\xi, t)$$

$$\frac{\partial^2 v}{\partial y^2} \longmapsto (2\pi i \xi)^2\, \hat{v}(\xi, t),$$

we have

$$\begin{cases} \left[(2\pi i \xi)^2 + (1-\alpha)(2\pi i \xi)\right] \hat{v}(\xi, t) = \frac{\partial}{\partial t}\hat{v}(\xi, t) \\ \hat{v}(\xi, 0) = \hat{F}(\xi) \end{cases}$$

By solving,

$$\hat{v}(\xi, t) = \hat{F}(\xi)\, e^{[-4\pi^2\xi^2 + (1-\alpha)2\pi i \xi]t}$$

$$= \hat{F}(\xi)\, e^{-4\pi^2 t \xi^2} \cdot e^{2\pi i [(1-\alpha)t]\xi}$$

Recall that $\hat{H}_t(\xi) = e^{-4\pi^2 t \xi^2}$ (See p. 146)

Let $g_t(x) := H_t(x + (1-\alpha)t)$

Prop 1.2 $\Rightarrow$ $\hat{v}(\xi, t) = \hat{F}(\xi)\, \hat{g}_t(\xi)$

Fourier inversion $\overset{\text{formula}}{\Longrightarrow}$ $v(y, t) = F * g_t(y)$

$$= \int_{-\infty}^{\infty} F(z)\, g_t(y-z)\, dz$$

$$= \int_{-\infty}^{\infty} F(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(y - z + (1-\alpha)t)^2} dz$$

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$$u(x,t) = U(y,t) \qquad (x = e^{-y})$$

$$= U(\log \tfrac{1}{x}, t)$$

$$= \int_{-\infty}^{\infty} F(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(\log \frac{1}{x} - z + (1-\alpha)t)^2} dz$$

$F(z) = f(e^{-z})$ Let $v = e^{-z}$, $dv = -e^{-z}\,dz$

$dz = -\frac{dv}{v}$

$z = -\log v$

$$= \int_{\infty}^{0} f(v) \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(\log \frac{1}{x} + \log v + (1-\alpha)t)^2} \frac{dv}{-v}$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{-(\log(v/x) + (1-\alpha)t)^2/4t} f(v) \frac{dv}{v}$$

3. Recall that the Fourier transform of $h(x) := e^{-|x|}\cos x$

is given by

$$\hat{h}(\xi) = \frac{2(2\pi\xi)^2 + 4}{(2\pi\xi)^4 + 4}$$

Hence, compute the integral

$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2} dx$$

3.

Note

$$\hat{h}(\xi) = \frac{2\,((2\pi\xi)^2+2)}{(2\pi\xi)^4+4}$$

If we put $g(x) = [2\pi\, h(2\pi x)]\,\frac{1}{2} = \pi h(2\pi x)$.

then $\hat{g}(\xi) = \frac{1}{2}\hat{h}\left(\frac{\xi}{2\pi}\right) = \frac{\xi^2+2}{\xi^4+4}$

The integral is $\int_{-\infty}^{\infty} (\hat{g}(\xi))^2\, d\xi$

$$= \int_{-\infty}^{\infty} \hat{\hat{g}}(\xi)\, g(\xi)\, d\xi$$

(multiplication formula)
Prop. 1.8 on p. 140

$$= \int_{-\infty}^{\infty} g(-x)\, g(x)\, dx$$

$$= \pi^2 \int_{-\infty}^{\infty} h(-2\pi x)\, h(2\pi x)\, dx$$

$$= \pi^2 \int_{-\infty}^{\infty} e^{-4\pi|x|} (\cos 2\pi x)^2\, dx$$

$$= \pi^2 \int_{-\infty}^{\infty} e^{-4\pi|x|} \left(\frac{e^{i2\pi x}+e^{-i2\pi x}}{2}\right)^2 dx$$

$$= \frac{\pi^2}{4} \int_{-\infty}^{\infty} e^{-4\pi|x|} \left(e^{-2\pi i(-2)x} + e^{-2\pi i(2)x} + 2\right) dx$$

$$= \frac{\pi^2}{4} \left(2\hat{f}(0) + \hat{f}(-2) + \hat{f}(2)\right); \quad f(x) = e^{-4\pi|x|}$$

Recall Lemma 2.4 on p. 456.

$$\Rightarrow \quad \hat{f}(\xi) = \frac{1}{\pi} \frac{2}{\xi^2+4}$$

$$\therefore \int_{-\infty}^{\infty} \hat{g}(\xi)^2 \, d\xi = \frac{\pi^2}{4} \left( 2 \cdot \frac{2}{\pi} \frac{1}{4} + \frac{1}{\pi} \frac{2}{8} + \frac{1}{\pi} \frac{2}{8} \right)$$

$$= \frac{3\pi}{8}$$

Last time

to show that

$$\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n^2+1} > 0 \qquad \forall\ x$$

Recall that

$$\pi e^{-2\pi|x|} \xrightarrow{\mathcal{F}} \frac{1}{1+\xi^2}$$

Let $f(x) = \pi e^{-2\pi|x|}$,

i.e. to show that

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{inx} > 0$$

But Poisson summation formula

$$\Rightarrow \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n(\frac{x}{2\pi})}$$

$$\textcircled{=} \sum_{n\in\mathbb{Z}} f(\tfrac{x}{2\pi} + n) > 0 \quad \because f > 0$$