3093 Tuto 9

- Fourier inversion formula (Ex 1)

(Recover f from its Fourier transform $\hat{f}$)

- Application of Prop. 1.2 to calculate Fourier transform

- $\widehat{f*g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$

- Show that $f*g$ is of moderate decrease if f, g are of moderate decrease (Ex 7)

Recall that for a cont. fcn f, we say f is of <u>moderate decrease</u> if $\exists A>0$ s.t.

$$|f(x)| \leq \frac{A}{1+x^2} \quad \forall x \in \mathbb{R}$$

- x^2 can be replaced by $x^{1+\varepsilon}$ for any $\varepsilon > 0$.

In particular,

$$\int_{-\infty}^{\infty} f(x) = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \quad \text{exists}$$

$$\int_{0}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{0}^{R} f(x)\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Ex 1 : Show the Fourier inversion formula if f is a continuous fcn with finite support $[-M, M]$ and $\hat{f}$ is of moderate decrease.

Step 1 : Let $L > 0$ s.t. $M < L/2$, show that

$$\left[f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{2\pi i x \left(\frac{n}{L}\right)} \quad \text{for } x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \right]$$

Let h be a L-periodic fcn s.t.

$$h(x) = f(x) \quad \text{on} \quad \left[-\frac{L}{2}, \frac{L}{2}\right]$$

Its Fourier coefficients

$$\hat{h}(n) = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-2\pi i n x/L} dx$$

$$= \frac{1}{L} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x/L} dx = \frac{1}{L} \hat{f}\left(\frac{n}{L}\right)$$

Fourier transform

It suffices to show that

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{L}\right)\right| < \infty \quad \left(\because \hat{f} \text{ is moderate decrease} \right)$$

Note $\displaystyle\sum_{n\neq 0} |\hat{f}(\tfrac{n}{L})| \leq \sum_{n\neq 0} \frac{A}{1+(\frac{n}{L})^2}$

$$= \sum_{n=1}^{\infty} \frac{2AL^2}{L^2+n^2}$$

$$\leq 2AL^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$\therefore$ $\displaystyle f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \hat{f}(\tfrac{n}{L})\, e^{2\pi i n x/L}$ ($\because$ f is cont) on $[-\frac{L}{2}, \frac{L}{2}]$

Step 2 : Suppose F is cont. and of moderate decrease, show that

$$\int_{-\infty}^{\infty} F(x)\,dx = \lim_{\delta\to 0^+} \sum_{n=-\infty}^{\infty} F(n\delta)\cdot\delta$$

Pf : $\displaystyle\int_{-\infty}^{\infty} F(x)\,dx = \lim_{N\to\infty}\int_{-N}^{N} F(x)\,dx$

$$\int_{-N}^{N} F(x)\,dx \approx \sum_{|n|\leq \frac{N}{\delta}} F(\delta n)\,\delta$$

Just consider the partition P

[assume $N_0\delta \le N < (N_0+1)\delta$]

$$P = \{-N, -N_0\delta, (-N_0+1)\delta, \ldots, -\delta, 0, \delta, 2\delta, \ldots, N_0\delta, N\}$$

We can tag the right end-points to obtain a Riemann sum

$$F(-N_0\delta)\cdot(N-N_0\delta) + \sum_{n=-N_0+1}^{N_0} F(n\delta)\,\delta + F(N)(N-N_0\delta)$$

$$= \sum_{n=-N_0}^{N_0} F(n\delta)\cdot\delta - F(-N_0\delta)(\overbrace{(N_0+1)\delta - N}^{<\delta}) + F(N)(\underbrace{N-N_0\delta}_{<\delta})$$

By bddness of F and Riemann integrability of F on $[-N,N]$,

$$\int_{-N}^{N} F(x)\,dx = \lim_{\delta\to 0^+} \sum_{|n|\le \frac{N}{\delta}} F(\delta n)\,\delta$$

Finally, we argue that

$\sum_{|n|>\frac{N}{\delta}} F(\delta n)\delta$ can be controlled merely by N

$$\sum_{|n|>\frac{N}{\delta}} |F(\delta n)\delta| \leq \sum_{|n|>\frac{N}{\delta}} \frac{A}{1+(\delta n)^2}\cdot\delta$$

$N_0 > \frac{N}{\delta} \geq N_0 - 1$:

$$\boxed{\sum_{n>\frac{N}{\delta}} \frac{1}{n^2}} = \sum_{n=N_0}^{\infty} \frac{1}{n^2} \leq \sum_{n=N_0}^{\infty} \int_{n-1}^{n} \frac{1}{x^2}\,dx$$

$$= \int_{N_0-1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N_0-1} \leq \boxed{\frac{1}{\frac{N}{\delta}}}$$

$$\leq \sum_{|n|>\boxed{\frac{N}{\delta}}} \frac{A\delta}{\delta^2 \boxed{n^2}}$$

$$\leq C\cdot\frac{1}{\delta}\cdot\frac{1}{\boxed{\frac{N}{\delta}}} = \frac{C}{N}$$

$$\int_{-\infty}^{\infty} F(x)\,dx \sim \int_{-N}^{N} F(x)\,dx \qquad N\to\infty$$

$$\int_{-N}^{N} F(x)\,dx \sim \sum_{|n|\leq\frac{N}{\delta}} F(\delta n)\delta \qquad \text{fixed } N,\ \delta\to 0$$

$$\sum_{|n|\leq\frac{N}{\delta}} F(\delta n)\delta \sim \sum_{n=-\infty}^{\infty} F(\delta n)\delta \qquad N\to\infty$$

Since $\hat{f}$ is of moderate decrease,

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi = \lim_{\delta \to 0^+} \sum_{n=-\infty}^{\infty} \delta \hat{f}(n\delta) e^{2\pi i n\delta x}$$

$$= f(x) \qquad \forall x \in [-\tfrac{L}{2}, \tfrac{L}{2}]$$

But L is any positive number with

$\frac{L}{2} > M$.

$$\therefore \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi \qquad \forall x \in \mathbb{R}$$

2. Find the Fourier transform of the following functions:

(a) $f(x) = e^{-x}H(x)$, where $H(x) := \chi_{[0,\infty)}$ is the Heaviside function.

(b) $g(x) = e^{-|x|}$.

(c) $h(x) = e^{-a|x|}$, $a > 0$.

(d) $k(x) = e^{-|x|}\cos x$.

Applying Prop 1.2 on p.136

(a).

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-x} H(x) e^{-2\pi i x \xi} dx$$

$$= \int_0^{\infty} e^{-x} e^{-2\pi i x \xi} dx$$

$$= \frac{1}{-1-2\pi i \xi} e^{-x-2\pi i x \xi}\Big|_{x=0}^{\infty}$$

$$= \frac{1}{1+2\pi i \xi}$$

(b)

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i x \xi} dx$$

$$= \int_0^{\infty} e^{-x} e^{-2\pi i x \xi} dx + \int_{-\infty}^{0} e^{x} e^{-2\pi i x \xi} dx$$

$$= \frac{1}{1+2\pi i \xi} + \frac{1}{1-2\pi i \xi}$$

$$= \frac{2}{1+4\pi^2\xi^2}$$

$$\left(f(\delta x) \xrightarrow{\mathcal{F}} \delta^{-1} \hat{f}(\delta^{-1}\xi) \quad \forall \delta > 0 \right)$$

(c).

$$h(x) = e^{-a|x|} \qquad (a > 0)$$

$$= g(ax)$$

$$\hat{h}(\xi) = \frac{1}{a}\hat{g}\left(\frac{\xi}{a}\right) = \frac{1}{a} \frac{2}{1+4\pi^2\left(\frac{\xi}{a}\right)^2}$$

$$= \frac{2a}{a^2+4\pi^2\xi^2}$$

(d)

$$k(x) = e^{-|x|}\cos x = e^{-|x|} \cdot \frac{e^{ix}+e^{-ix}}{2}$$

$$= g(x) \cdot \frac{e^{-2\pi i x\left(\frac{-1}{2\pi}\right)} + e^{-2\pi i x\left(\frac{1}{2\pi}\right)}}{2}$$

$$\hat{k}(\xi) = \frac{\hat{g}\left(\xi - \frac{1}{2\pi}\right) + \hat{g}\left(\xi + \frac{1}{2\pi}\right)}{2}$$

$$\left(\because \quad f(x)\, e^{-2\pi i x h} \xrightarrow{\mathcal{F}} \hat{f}(\xi + h) \quad \forall h \in \mathbb{R} \right)$$

$$= \frac{\frac{2}{1+4\pi^2\left(\xi - \frac{1}{2\pi}\right)^2} + \frac{2}{1+4\pi^2\left(\xi + \frac{1}{2\pi}\right)^2}}{2}$$

$$= \frac{1}{1+(2\pi\xi-1)^2} + \frac{1}{1+(2\pi\xi+1)^2}$$

3. Find f that satisfies the integral equation

$$\int_{-\infty}^{\infty} f(x-y)e^{-|y|}dy = 2e^{-|x|} - e^{-2|x|}.$$

(**Hint:** Apply Fourier transform and the properties of convolution.)

$$LHS = f * g(x) \quad , \quad g(y) := e^{-|y|}$$

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

[Prop 1.11 on p 142
See also remark 1.7 on p.144]

$$\therefore \quad \hat{f}(\xi)\hat{g}(\xi) = \int_{-\infty}^{\infty} \left(2e^{-|x|} - e^{-2|x|}\right) e^{-2\pi i x\xi} dx$$

$$= \frac{4}{1+4\pi^2\xi^2} - \frac{4}{4+4\pi^2\xi^2} \quad (\text{by Q2})$$

$$= \frac{4}{1+4\pi^2\xi^2}\left(1 - \frac{1+4\pi^2\xi^2}{4+4\pi^2\xi^2}\right)$$

$$= \frac{2}{1+4\pi^2\xi^2} \cdot \frac{6}{4+4\pi^2\xi^2}$$

$$= \hat{g}(\xi)\left[\frac{3}{2}\hat{h}(\xi)\right]$$

where $h(x) = e^{-2|x|}$

$$\therefore \quad \hat{f}(\xi) = \frac{3}{2}\hat{h}(\xi)$$

If f is continuous fcn of moderate decrease, then Fourier inversion formula forces $f(x) = \frac{3}{2}e^{-2|x|}$

Now, if we put $f(x) = \frac{3}{2}e^{-2|x|}$,

then

$f*g(x)$ and $2e^{-|x|} - e^{-2|x|}$

have the same Fourier transform $\frac{2}{1+4\pi^2\xi^2}\frac{6}{4+4\pi^2\xi^2}$

Since $f*g$, $2e^{-|x|} - e^{-2|x|}$, and $\frac{2}{1+4\pi^2\xi^2}\frac{6}{4+4\pi^2\xi^2}$

are continuous fcn of moderate decrease,

we can apply Fourier inversion formula

to conclude that

$$f*g(x) = 2e^{-|x|} - e^{-2|x|}.$$

Hence $f(x) = \frac{3}{2}e^{-2|x|}$ is a solution.

Ex 4. Convolution of two fcns of moderate decrease is also a fcn of moderate decrease. (Ex 7 in textbook)

Pf:

$$f*g(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

$$= \int_{|y| \le \frac{|x|}{2}} f(x-y)\,g(y)\,dy + \int_{|y| \ge \frac{|x|}{2}} f(x-y)\,g(y)\,dy$$

For $\int_{|y| \le \frac{|x|}{2}}$, $|x|$ is larger, we use

$$f(x) = O\left(\frac{1}{1+x^2}\right)$$

hence

$$\int_{|y|\le\frac{|x|}{2}} |f(x-y)g(y)|\,dy \le \int_{|y|\le\frac{|x|}{2}} \frac{A}{1+(x-y)^2}|g(y)|\,dy$$

$$|x-y| \ge |x|-|y|$$
$$\ge |x|-\frac{|x|}{2} = \frac{|x|}{2} \qquad \text{if } |y|\le\frac{|x|}{2}$$

$$\Rightarrow \quad \frac{A}{1+(x-y)^2} \le \frac{A}{1+\left(\frac{x}{2}\right)^2}$$

$$\therefore \int_{|y|\le\frac{|x|}{2}} \le \frac{A}{1+\left(\frac{x}{2}\right)^2}\int_{|y|\le\frac{|x|}{2}} |g(y)|\,dy$$

$$\le \frac{A}{1+\left(\frac{x}{2}\right)^2}\int_{-\infty}^{\infty} |g(y)|\,dy$$

$\int_{|y|\ge\frac{|x|}{2}}$: Use $g(x) = O\left(\frac{1}{1+x^2}\right)$

Exercise.