Let $\mathcal{F} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ be Fourier transform. Aim to generate some eigenfunctions of $\mathcal{F}$ using $\psi(x) = e^{-\lambda x^2}$.

(Ch 5. Ex 6)

Eg. $(\mathcal{F}\psi)(x) = \psi(x)$

$\therefore$ ψ is an eigenfunction w.r.t. eigenvalue 1.

$\vdash_1$: All possible eigenvalues of $\mathcal{F}$ are $\{\pm 1, \pm i\}$

Pf $\vdash_1$: By Fourier inversion formula,

$$(\mathcal{F} \circ \mathcal{F} f)(x) = f(-x) \qquad \forall f \in \mathcal{S}(\mathbb{R}),\ x \in \mathbb{R}$$

$$\Rightarrow \quad \mathcal{F}^4 f(x) = f(x)$$

i.e. $\quad \mathcal{F}^4 = I \quad$: the identity map on $\mathcal{S}(\mathbb{R})$

Suppose $\mathcal{F} f = \lambda f$ for some $\lambda \in \mathbb{C}$, $f \neq 0 \in S(\mathbb{R})$

then $f = \mathcal{F}^4 f = \lambda^4 f$

$\Rightarrow \quad 1 = \lambda^4$

$\therefore \quad \lambda \in \{\pm 1, \pm i\}$ $\#$

For $\lambda \in \{\pm 1, \pm i\}$,

$x^n - \lambda^n = (x-\lambda)(x^{n-1} + \lambda x^{n-2} + \dots + \lambda^{n-1})$

$$0 = \mathcal{F}^4 - I = \mathcal{F}^4 - \lambda^4 I$$
$$= (\mathcal{F} - \lambda)(\mathcal{F}^3 + \lambda \mathcal{F}^2 + \lambda^2 \mathcal{F} + \lambda^3 I)$$

Multiplying λ

$$\Rightarrow \quad 0 = (\mathcal{F} - \lambda)\left[(I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3) f\right]$$

belongs to null space of $\mathcal{F} - \lambda$

$\therefore \forall f \in S(\mathbb{R})$,

$$(I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3) f \in E_\lambda$$

$:=$ eigenspace of $\mathcal{F}$ w.r.t eigenvalue λ.

Notation: For each $f \in S(\mathbb{R})$, $\lambda \in \{\pm 1, \pm i\}$

put $f_\lambda = \frac{1}{4}(I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3) f$

$\in E_\lambda$

- $f = f_1 + f_{-1} + f_i + f_{-i}$

Idea: put $g(x) = (-2\pi i x)^n \varphi(x)$ $\left(\varphi(x) = e^{-\pi x^2}\right)$

then for some $\lambda \in \{\pm 1, \pm i\}$ depending on n,

g_λ would be "meaningful" eigenfunction.

Note:

① $g(x) = (-2\pi i x)^n \varphi(x)$

$(\mathcal{F} g)(x) = \varphi^{(n)}(x)$

$(-2\pi i x) f(x) \xmapsto{\mathcal{F}} \hat{f}'(\xi)$ $\quad \hat{f}''(\xi)$

$$(-2\pi i x)^2 f(x) \xmapsto{\mathcal{F}} \frac{d}{d\xi} \underbrace{\mathcal{F}[-2\pi i x f(x)]}_{= \hat{f}'(\xi)}(\xi)$$

② $\mathcal{F}^2 f(x) = f(-x) \quad \forall f \in S(\mathbb{R})$

③ $g(-x) = (-1)^n (-2\pi i x)^n \psi(-x)$

$= (-1)^n g(x) \quad$ (ψ is even)

④ $\mathcal{F} g(-x) = \psi^{(n)}(-x) = \psi^{(n)}(x) (-1)^n$

Derivative of even fcn is odd

" " odd " even

Example:

$\lambda = 1$:

$$g_1(x) = \frac{1}{4} \left[(I + \mathcal{F} + \mathcal{F}^2 + \mathcal{F}^3) g \right](x)$$

$$= \frac{1}{4} \left[g(x) + \mathcal{F}g(x) + g(-x) + \mathcal{F}g(-x) \right]$$

$$= \frac{1}{4} \left[g(x) + (-1)^n g(x) + \psi^{(n)}(x) + (-1)^n \psi^{(n)}(x) \right]$$

So, simple observation

$g_1(x) = 0$ if n is odd

After some inspection, for each $n \in \mathbb{N}$ and $g(x) = (-2\pi i x)^n \psi(x)$, we should choose $\lambda = (-i)^n$, so that $g_\lambda(x) = p_n(x)\psi(x) \in E_\lambda$ is nonzero, and $p_n(x)$ is a polynomial of degree $= n$

Indeed,

$$g_\lambda(x) = \left(\frac{1}{4}\left[I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3\right] g\right)(x)$$

$$= \frac{1}{4}\left(g(x) + (-i)^{3n} \psi^{(n)}(x) + (-i)^{2n} (-1)^n g(x) + (-i)^n (-1)^n \psi^{(n)}(x) \right)$$

$$= \frac{1}{4}\left(2g(x) + 2(i)^n \psi^{(n)}(x) \right)$$

$$= \frac{1}{2}\left(g(x) + (i)^n \psi^{(n)}(x) \right)$$

$$\psi^{(n)}(x) = \frac{d^n}{dx^n}\left(e^{-\pi x^2}\right) = e^{-\pi x^2}\left[(-2\pi x)^n + Q_n(x)\right]$$

$\deg Q_n \leq n-1$

$$= \frac{1}{2}\left((-2\pi i x)^n \psi(x) + \psi(x)(i)^n\left[(-2\pi x)^n + Q_n(x)\right]\right.$$

$$= \frac{1}{2}\left[2(-2\pi i x)^n \psi(x) + (i)^n Q_n(x)\psi(x)\right]$$

$$= p_n(x)\psi(x) \quad , \quad \deg p_n = n.$$

To check whether a sequence $\{a_n\}_{-\infty}^{\infty}$ can be Fourier coefficients of some $f \in R[-\pi,\pi]$

① $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$

② Abel mean : (p, 83-84)

if $a_n \geq 0 \ \forall n$, then

$\{a_n\}$ can be Fourier coefficient

only if

$$\sum_{n=-\infty}^{\infty} a_n < \infty$$