MATH 3093 Fourier Analysis

Tutorial 7 (March 16)

The following will be discussed in the tutorial this week:

1. Recall that a sequence of numbers $\{x_n\}$ in $[0,1)$ is said to be *equidistributed* if for any interval $I \subset [0,1)$,

$$\frac{1}{N}\sum_{n=1}^{N}\chi_I(x_n) \to |I| \quad \text{as } N \to \infty,$$

   where $|I|$ denote the length of I.

2. We have the following characterization of equidistributed sequence:

   Theorem (Weyl)**.** *Let $\{x_n\}$ be a sequence of real numbers in $[0,1)$. Then the following are equivalent:*

   (a) $\{x_n\}$ is equidistributed;

   (b) for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \to 0;$$

   (c) for any 1-periodic continuous function f,

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) \to \int_0^1 f(y)dy.$$

3. Let $a \neq 0$ and $0 < \sigma < 1$. Show the the sequence $\langle an^\sigma \rangle$ is equidistributed in $[0,1)$. Here $\langle x \rangle$ denotes the fractional part of x.

4. Prove that $\langle a \log n \rangle$ is not equidistributed for any a.