MATH 3093 Fourier Analysis

Tutorial 6 (March 9)

The following will be discussed in the tutorial this week:

1. (Wirtinger's inequality) Let f be a T-periodic, continuous and piecewise C^1 function.

   (a) Suppose
   $$\int_0^T f(t)dt = 0.$$
   Show that
   $$\int_0^T |f(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2 dt,$$
   with equality holds if and only if f takes the form $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$.

   (b) Hence concludes that, in general,
   $$\int_0^T |f(t) - \bar{f}|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2 dt,$$
   where $\bar{f} = \frac{1}{T}\int_0^T f(t)dt$. When does equality hold in this case?

2. Let f, g be T-periodic, continuous and piecewise C^1 functions. Suppose
   $$\int_0^T f(t)dt = 0.$$
   Show that
   $$\left|\int_0^T \overline{f(t)}g(t)dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2 dt \int_0^T |g'(t)|^2 dt.$$

3. Let f be a C^1 function on $[a, b]$ such that $f(a) = f(b) = 0$. Show that
   $$\int_a^b |f(t)|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2 dt.$$
   When does equality hold?