MATH 3093 Fourier Analysis

Tutorial 3 (February 17)

The following will be discussed in the tutorial this week:

1. (Cesàro summability) Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers. Set

$$s_n := \sum_{k=1}^{n} c_k \quad \text{and} \quad \sigma_N := \frac{1}{N}\left(s_1 + \cdots s_N\right).$$

We say that $\sum c_n$ is Cesàro summable to $\sigma (\in \mathbb{C})$ if $\sigma_N \to \sigma$ as $N \to \infty$.

2. (Abel summability) Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers. For any $0 \le r < 1$, set

$$A(r) = \sum_{n=1}^{\infty} c_n r^n.$$

We say that $\sum c_n$ is Abel summable to $s (\in \mathbb{C})$ if $A(r)$ converges for every $r \in [0, 1)$ and

$$\lim_{r \to 1^-} A(r) = s.$$

3. Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers. Show that

$$\sum c_n \text{ convergent } \Rightarrow \sum c_n \text{ Cesàro summable } \Rightarrow \sum c_n \text{ Abel summable},$$

and none of the implications can be reversed.

4. Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers. Suppose $nc_n \to 0$ as $n \to \infty$. Show that

$$\sum c_n \text{ convergent } \Leftrightarrow \sum c_n \text{ Cesàro summable } \Leftrightarrow \sum c_n \text{ Abel summable}.$$

(**Hint:** Consider $\sum_{n=1}^{N} c_n - A(1 - 1/N)$ for large N.)