MATH 3093 Fourier Analysis

Tutorial 2 (January 20)

The first two items were discussed in the tutorial this week:

1. Let f be a Riemann integrable function on $[-\pi, \pi]$. Then the Fourier series of f is given by
$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx},$$
where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy$. Equivalently
$$f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$
where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy$ for $n \geq 1$. Find the relation between $\hat{f}(n)$ and a_n, b_n.

2. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} -\pi/2 - x/2 & \text{if } -\pi \leq x < 0, \\ 0 & \text{if } x = 0, \\ \pi/2 - x/2 & \text{if } 0 < x \leq \pi. \end{cases}$$
   (a) Compute the Fourier coefficients $\hat{f}(n)$, a_n and b_n of f. Verify that they do satisfy the relation obtained in Q1.
   (b) Show that the Fourier series of f converges at every point $x \in [-\pi, \pi]$. (**Hint:** you may apply the Dirichlet's test.)

3. We say that the Fourier series of f converges absolutely if
$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$
Let $-\pi < a < b < \pi$. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the characteristic function of $[a, b]$, that is
$$f(x) = \chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Find the Fourier series of f.
   (b) Show that the Fourier series of f converges at every $x \in [-\pi, \pi]$.
   (c) Show that the Fourier series of f is NOT absolutely convergent. (**Hint:** Let $\theta_0 = (b - a)/2$. Note $\theta_0 \in (0, \pi)$. Find an interval $[c, \pi - c] \subset (0, \pi)$ that has length $> \theta_0$. Show that for any $k \geq 1$, the interval $[k\pi + c, (k + 1)\pi - c]$ contains $n_k \theta_0$ for some $n_k \geq 1$.)