Cesàro summability and Abel summability

MATH3093 Fourier Analysis Tutorial 3

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Introduction

- It is well-known that the sequence $\{0, 1, 0, 1, \ldots\}$ does not converge, but we may think that $\frac{1}{2}$ is somehow a "limit" of the sequence.
- Let x_n be the above sequence. Then, $\frac{1}{2}$ is the limit of the sequence (y_n), where $y_n = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$, the average of the first n-term.
- Remark: The "limit" is not affected by finitely many terms.

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- Remark: The "limit" is not affected by finitely many terms.
- This can also apply to series.

Cesàro summability

- Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers.
- Let $s_n = c_1 + c_2 + \ldots + c_n$ be its partial sums.
- We say that the series converges iff (s_n) converges.

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- Let $s_n = c_1 + c_2 + \ldots + c_n$ be its partial sums.
- We say that the series converges iff (s_n) converges.
- We say that the series is Cesàro summable if $\sigma_N = \dfrac{s_1 + s_2 + \ldots + s_N}{N}$ converges.

Example

- We may put $(c_n) = \{1, -1, 1, -1, \ldots\}$, then $(s_n) = \{1, 0, 1, 0, \ldots\}$.
- By slide 2, the series $\sum_{n=1}^{\infty} c_n$ is not convergent, but it is Cesàro summable to $\frac{1}{2}$.

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- We may put $(c_n) = \{1, -1, 1, -1, \ldots\}$, then $(s_n) = \{1, 0, 1, 0, \ldots\}$.
- By slide 2, the series $\sum_{n=1}^{\infty} c_n$ is not convergent, but it is Cesàro summable to $\frac{1}{2}$.
- It is a standard exercise in MATH2050 that: if a sequence (x_n) converges to the limit x, then

$$y_n := \frac{x_1 + x_2 + \ldots + x_n}{n}$$

converges to the same limit x. Therefore, every convergent series is Cesàro summable.

Abel summability

We say that a series $\sum_{n=1}^{\infty} c_n$ is Abel summable to s if the following holds.

(i) For every $0 < r < 1$,

$$A(r) := \sum_{n=1}^{\infty} c_n r^n$$

converges.

(ii) $\lim_{r \to 1^-} A(r) = s$.

Abel summability

We want to show that every convergent series is Abel summable to the same limit. Let $s = \sum_{n=1}^{\infty} c_n$.

- In MATH2060, we learnt that if a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x = x_0$, where x_0 may not a real number, then the power series also converges for $|x| < |x_0|$.
- Moreover, it converges uniformly on $\{x \in \mathbb{C} : |x| < \alpha\}$ for every $\alpha < |x_0|$.

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- Moreover, it converges uniformly on $\{x \in \mathbb{C} : |x| < \alpha\}$ for every $\alpha < |x_0|$.
- This shows that $A(r)$ must exist for $0 < r < 1$, when the series converges.

Abel summability

To show that $\lim_{r \to 1^-} A(r) = s$, we need some calculations.

$$\text{Step 1:} \quad \sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n \quad \text{for } 0 < r < 1$$

$$\text{Step 2:} \quad s = \lim_{r \to 1^-} (1-r) \sum_{n=1}^{\infty} s r^n$$

$$\text{Step 3:} \quad \lim_{r \to 1^-} (1-r) \sum_{n=1}^{\infty} (s_n - s) r^n = 0 \quad \because \lim_{n \to \infty} s_n = s$$

Two questions

1. Q3: Cesàro summability implies Abel summability
2. Q4: Tauberian theorem: conditions when Abel summability implies series convergence

Cesàro summability implies Abel summability

- We assume that $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to s, and we want to show that the series is also Abel summable to s.
- Recall that $s_n = c_1 + c_2 + \ldots + c_n$ and

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n}.$$

- Cesàro summability is the convergence of σ_n. We would like to rewrite

$$A(r) = \sum_{n=1}^{\infty} c_n r^n$$

in terms of σ_n.

Rewrite $A(r)$ in terms of σ_n

Notice that $c_n = s_n - s_{n-1}$, $(s_0 = 0)$. From which, we have

$$\begin{aligned}
\sum_{n=1}^{N} c_n r^n &= \sum_{n=1}^{N} (s_n - s_{n-1}) r^n \\
&= \sum_{n=1}^{N} s_n r^n - \sum_{n=0}^{N-1} s_n r^{n+1} \\
&= \sum_{n=1}^{N-1} s_n (r^n - r^{n+1}) + s_N r^N \\
&= (1-r) \sum_{n=1}^{N-1} s_n r^n + s_N r^N
\end{aligned}$$

Rewrite $A(r)$ in terms of σ_n

Similarly, note that $s_n = n\sigma_n - (n-1)\sigma_{n-1}$, $(\sigma_0 = 0)$, thus we have

$$\sum_{n=1}^{N-1} s_n r^n = (1-r)\sum_{n=1}^{N-2} n\sigma_n r^n + (N-1)\sigma_{N-1} r^{N-1}$$

Combining with the slide above, we have

$$\sum_{n=1}^{N} c_n r^n = (1-r)^2 \sum_{n=1}^{N-2} n\sigma_n r^n + (1-r)(N-1)\sigma_{N-1} r^{N-1} + s_N r^N$$

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For any fixed $0 < r < 1$, the last two terms go to 0 when $N \to \infty$. This follows from

$$\lim_{n\to\infty} nr^n = 0 \quad \text{(Ratio text)}$$

Rewrite $A(r)$ in terms of σ_n

- Therefore,

$$A(r) = \sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

  whenever exists.

- This exists for $0 < r < 1$.

- Since (σ_n) is bounded, say by M, we have

$$\limsup \sqrt[n]{|n\sigma_n|} \leq \limsup \sqrt[n]{n}\sqrt[n]{M} \leq 1.$$

- The radius of convergence is given by $\dfrac{1}{\limsup \sqrt[n]{|n\sigma_n|}}$.

Cesàro summability implies Abel summability

- Recall that

$$A(r) = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n.$$

- By the well-known formula $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$ for $0 < r < 1$, we have

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

- Therefore, $s = \lim_{r \to 1^-} (1-r)^2 \sum_{n=1}^{\infty} nsr^n$.

Cesàro summability implies Abel summability

- Compare $(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$ and $(1-r)^2 \sum_{n=1}^{\infty} nsr^n$.
- Their difference is
$$(1-r)^2 \sum_{n=1}^{\infty} n(\sigma_n - s)r^n$$

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- For the first N terms, whenever N is fixed, they can be well-controlled by $r \to 1^-$.

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- Their difference is
$$(1-r)^2\sum_{n=1}^{\infty} n(\sigma_n - s)r^n$$
- For the first N terms, whenever N is fixed, they can be well-controlled by $r\to 1^-$.
- We may choose N large so that $|\sigma_n - s| < \epsilon$ for $n > N$.
$$\left|(1-r)^2\sum_{n=N+1}^{\infty} n(\sigma_n - s)r^n\right| \le (1-r)^2\epsilon\sum_{n=N+1}^{\infty} nr^n \le \epsilon.$$

Abel summability is strictly weaker than Cesàro

- We may construct a series $\sum_{n=1}^{\infty} c_n$ such that $(\sigma_n) = \{0, 1, 0, 1, \ldots\}$. Then, this series is not Cesàro summable.

- In this case,

$$A(r) = (1-r)^2 \sum_{n=1}^{\infty} 2nr^{2n} = (1-r)^2 \frac{2r^2}{(1-r^2)^2} = \frac{2r^2}{(1+r)^2}$$

- The series is Abel summable to $\lim_{r \to 1^-} A(r) = \frac{1}{2}$

Abel summability is strictly weaker than Cesàro

- $(\sigma_n) = \{0, 1, 0, 1, \ldots\}$.
- Recall that

$$s_n = n\sigma_n - (n-1)\sigma_{n-1} = \begin{cases} n & \text{if } n \text{ is even;} \\ -(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

- $(s_n) = \{0, 2, -2, 4, -4, \ldots\}$.
- Recall that

$$c_n = s_n - s_{n-1} = \begin{cases} 2n-2 & \text{if } n \text{ is even;} \\ -2(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

- $(c_n) = \{0, 2, -4, 6, -8 \ldots\}$.

Tauberian theorem

- Let $\sum_{n=1}^{\infty} c_n$ be a series such that $nc_n \to 0$ as $n \to \infty$. If the series is Abel summable, then it converges. $(c_n = o(\frac{1}{n}))$
- Hint: consider $\sum_{n=1}^{N} c_n - A(1 - \frac{1}{N})$.
  Suffices to show that it converges to zero.
- The proof is similar to that of **Lemma 2.3** in our textbook.

Tauberian theorem

- Let $r = 1 - \frac{1}{N}$.

$$\sum_{n=1}^{N} c_n - A(r) = \sum_{n=1}^{N} c_n(1 - r^n) - \sum_{n=N+1}^{\infty} c_n r^n$$

- Note that $1 - r^n = (1 - r)(1 + r + r^2 + \ldots + r^{n-1}) \leq \frac{n}{N}$. This helps approximate the first term.

- Indeed,

$$\left| \sum_{n=1}^{N} c_n(1 - r^n) \right| \leq \frac{1}{N} \sum_{n=1}^{N} n|c_n|$$

So, the first term goes to zero when $N \to \infty$. ($c_n = o(\frac{1}{n})$)

Tauberian theorem

- Let $r = 1 - \frac{1}{N}$.

$$\sum_{n=1}^{N} c_n - A(r) = \sum_{n=1}^{N} c_n(1 - r^n) - \sum_{n=N+1}^{\infty} c_n r^n$$

- Let $\epsilon > 0$, since $c_n = o(\frac{1}{n})$, for some $N_0 \in \mathbb{N}$, we have $|c_n| < \frac{\epsilon}{n}$ for any $n \geq N_0$. Note then for $N \geq N_0$,

$$\left| \sum_{n=N+1}^{\infty} c_n r^n \right| \leq \sum_{n=N+1}^{\infty} \frac{\epsilon}{N} r^n = \frac{\epsilon}{N} \frac{r^{N+1}}{1 - r} \leq \epsilon$$

Reference

You may refer to Exercises 12, 13, 14 of Chapter 2.

Tutorial 2

We say that the Fourier series of f converges absolutely if

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Let $-\pi < a < b < \pi$. Let $f : [-\pi, \pi] \to$ be the characteristic function of $[a, b]$, that is

$$f(x) = \chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the Fourier series of f.

(b) Show that the Fourier series of f converges at every $x \in [-\pi, \pi]$.

(c) Show that the Fourier series of f is NOT absolutely convergent.

Tutorial 2

- Recall that

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x,$$

  and the Fourier series is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

- For part (a), by calculations, the answer is

$$\frac{b-a}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi n}(\sin n(b-x) - \sin n(a-x))$$

- Part (b) follows from Dirichlet's test.