Fourier Analysis Feb. 24.

Review.

Let V be an inner product space over $\mathbb{C}$. Then

① (Pythagorean Thm) If $x, y \in V$ with $\langle x, y\rangle = 0$, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

② (Cauchy-Schwartz)

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|.$$

③ (triangle inequality)

$$\|x+y\| \le \|x\| + \|y\|.$$

§.3.3 Proof of mean square convergence

Let $\mathcal{R} = \mathcal{R}[-\pi, \pi]$ be the space of $\mathbb{C}$-valued integrable functions on the circle. For $f, g \in \mathcal{R}$, we define

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot \overline{g(x)}\, dx$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{R}$ over $\mathbb{C}$.

For $n \in \mathbb{Z}$, write

$$e_n(x) = e^{inx}, \quad x \in [-\pi, \pi].$$

Then $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal in the sense that

$$\langle e_n, e_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

Furthermore for $f \in \mathcal{R}$ and $n \in \mathbb{Z}$,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \langle f, e_n \rangle .$$

For any trigonometric polynomial $\sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} c_n e_n$

$$\left\| \sum_{n=-N}^{N} c_n e_n \right\|^2 = \sum_{-N \leq n, m \leq N} c_n \overline{c_m} \langle e_n, e_m \rangle$$

$$= \sum_{n=-N}^{N} |c_n|^2 .$$

Lemma 1. For any $f \in \mathcal{R}$ and $N \in \mathbb{N}$, we have

$$(f - S_N f) \perp e_n \quad \text{for all} \quad |n| \leq N .$$

Pf. $\langle f - S_N f, e_n \rangle = \langle f, e_n \rangle - \langle S_N f, e_n \rangle$

$$= \hat{f}(n) - \widehat{S_N f}(n)$$

$$= \hat{f}(n) - \hat{f}(n) \quad (\text{for } |n| \leq N)$$

$$= 0 .$$

$\blacksquare$

Lemma 2 (Best approximation) Let $f \in \mathcal{R}$ and $N \in \mathbb{N}$,

Then

① $$\| f - S_N f \| \leq \| f - \sum_{n=-N}^{N} c_n e_n \|$$

for all $\{c_n\}_{n=-N}^{N} \subset \mathbb{C}$

② The "$=$" holds iff $c_n = \hat{f}(n)$ for $|n| \leq N$.

Proof. Given $\{c_n\}_{n=-N}^{N} \subset \mathbb{C}$,

$$f - \sum_{n=-N}^{N} c_n e_n = (f - S_N f) + (S_N f - \sum_{n=-N}^{N} c_n e_n).$$

By Lemma 1, $(S_N f - \sum_{n=-N}^{N} c_n e_n) \perp (f - S_N f)$,

hence by Pythagorean Thm,

$$\| f - \sum_{n=-N}^{N} c_n e_n \|^2 = \| f - S_N f \|^2 + \| S_N f - \sum_{n=-N}^{N} c_n e_n \|^2,$$

from which we see that

$$\| f - S_N f \| \leq \| f - \sum_{n=-N}^{N} c_n e_n \|,$$

and the "$=$" holds iff $\| S_N f - \sum_{n=-N}^{N} c_n e_n \|^2 = 0$.

But $$\| S_N f - \sum_{n=-N}^{N} c_n e_n \|$$

$$= \| \sum_{n=-N}^{N} (\hat{f}(n) - c_n) e_n \|^2$$

$$= \sum_{n=-N}^{N} | \hat{f}(n) - c_n |^2 = 0 \iff c_n = \hat{f}(n).$$

$\boxtimes$

Thm 3. Let $f \in R$. Then

(a) $\lim_{N\to\infty} \| f - S_N f \| = 0$.

(b) Parseval identity

$$\| f \|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Pf. We first prove (a).

Let $\varepsilon > 0$. We claim that $\exists$ a trigonometric polynomial P such that

$$\| f - P \| < \varepsilon.$$

Assume first that f is cts on the circle. In such case by Weierstrass approximation Thm, $\exists$ a trigonometric polynomial P such that

$$| f(x) - P(x) | < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Then

$$\| f - P \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - P(x)|^2 dx$$

$$< \varepsilon^2$$

so $\| f - P \| < \varepsilon$.

Next assume that f is integrable. As was proved before, $\exists$ a cts function g on the circle such that

① $\sup_{x} |g(x)| \leq \sup_{x\in[\pi,\pi]} |f(x)| =: \|f\|_\infty$

② $$\int_{-\pi}^{\pi} |f(x) - g(x)|dx \leq \frac{\varepsilon^2}{4\cdot\|f\|_\infty}$$

Pick a trigonometric poly P s.t.

$$\| g - P\| \leq \varepsilon/2.$$

Notice that

$$\begin{aligned}\|f-g\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx \\ &\leq \frac{1}{2\pi}\cdot 2\cdot\|f\|_\infty \int_{-\pi}^{\pi} |f(x)-g(x)|dx \\ &\leq \frac{1}{2\pi}\cdot 2\cdot\|f\|_\infty \cdot \frac{\varepsilon^2}{4\|f\|_\infty} \\ &\leq \frac{\varepsilon^2}{4\pi},\end{aligned}$$

So $\|f-g\| \leq \sqrt{\frac{\varepsilon^2}{4\pi}} < \frac{\varepsilon}{2}$.

Hence

$$\|f-P\| \leq \|f-g\| + \|g-P\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the claim.

Write $M = \deg(P)$.

For $N \geq M$, by the Best approximation Thm,

$$\|f - S_N f\| \leq \|f-P\| \quad (\text{since } P \text{ is a trigonometric poly with degree} \leq N)$$
$$< \varepsilon.$$

Hence

$$\lim_{N\to\infty} \|f - S_N f\| = 0$$

This proves (a).

Next we prove (b). Notice that by Lemma 1,

$$(f - S_N f) \perp S_N f.$$

By Pythagorean Thm, we have

$$\|f\|^2 = \|f - S_N f\|^2 + \|S_N f\|^2$$
$$= \|f - S_N f\|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

Letting $N \to \infty$, we have $\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$.