MATH 3093 Fourier Analysis

Tutorial 9 (April 6)

The following will be discussed in the tutorial this week:

1. (Simplified version of the Fourier inversion formula)
   Let f be a continuous function supported on $[-M, M]$, whose Fourier transform $\hat{f}$ is of moderate decrease. We would like to show that
   $$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$
   (a) Fix L with $L/2 > M$. Show that
   $$f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{f}(n/L) e^{2\pi i n x/L}$$
   for any $x \in [-L/2, L/2]$.
   (b) Prove that if F is continuous and of moderate decrease, then
   $$\int_{-\infty}^{\infty} F(\xi)\, d\xi = \lim_{\delta \to 0^+} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$
   (c) Prove the Fourier inversion formula.
2. Find the Fourier transform of the following functions:
   (a) $f(x) = e^{-x} H(x)$, where $H(x) := \chi_{[0,\infty)}$ is the Heaviside function.
   (b) $g(x) = e^{-|x|}$.
   (c) $h(x) = e^{-a|x|}$, $a > 0$.
   (d) $k(x) = e^{-|x|} \cos x$.
3. Find f that satisfies the integral equation
   $$\int_{-\infty}^{\infty} f(x-y) e^{-|y|} dy = 2e^{-|x|} - e^{-2|x|}.$$
   (**Hint:** Apply Fourier transform and the properties of convolution.)
4. Prove that the convolution of two functions of moderate decrease is a function of moderate decrease. [Hint: Write
   $$\int f(x-y) g(y)\, dy = \int_{|y| \le |x|/2} + \int_{|y| \ge |x|/2}.$$
   In the first integral $f(x-y) = O(1/(1+x^2))$ while in the second integral $g(y) = O(1/(1+x^2))$.]