MATH 3093 Fourier Analysis

Tutorial 8 (March 23)

The following were discussed in the tutorial this week:

1. Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Let $S_N = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Show that for any $H \leq N$, one has
$$|S_N|^2 \leq C \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \right|,$$
for some constant $C > 0$ independent of H, N and f.

2. Show that the sequence $\langle \gamma n^2 \rangle$ is equidistributed in $[0, 1)$ whenever γ is irrational.

3. More generally, show that if $\{x_n\}$ is a sequence in $[0, 1)$ such that for all positive integers h the difference $\langle x_{n+h} - x_n \rangle$ is equidistributed in $[0, 1)$, then $\{x_n\}$ is equidistributed in $[0, 1)$.

4. Let $P(x) = c_N x^N + \cdots + c_0$ be a polynomial with real coefficients. Suppose at least one of $c_1, \ldots, c_N$ is irrational. Show that the sequence $\langle P(n) \rangle$ is equidistributed in $[0, 1)$.

   [**Hint:** Argue by induction on the highest degree term which has an irrational coefficient. To prove the base case, consider $P(x) = Q(x) + \gamma x + c$, where γ is irrational and $Q(x)$ is a polynomial with rational coefficients. Find an integer L such that $LQ(x)$ is an integral polynomial. For any $1 \leq n \leq N$, write $n = kL + d$ and show that
$$Q(kL + d) = Q(d) + \text{ an integer.}]$$

Solution

3 More generally, show that if $\{x_n\}$ is a sequence in $[0, 1)$ such that for all positive integers h the difference $\langle x_{n+h} - x_n\rangle$ is equidistributed in $[0, 1)$, then $\{x_n\}$ is equidistributed in $[0, 1)$.

Proof. Fix $k \in \mathbb{Z} \setminus \{0\}$. Take $f(n) := kx_n$. Then for any $H \leq N$,

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n}\right|^2 &\leq \frac{C}{HN}\sum_{h=0}^{H-1}\left|\sum_{n=1}^{N-h} e^{2\pi i k(x_{n+h}-x_n)}\right| \\ &\leq \frac{C}{H} + \frac{C}{H}\sum_{h=1}^{H-1}\left(\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(x_{n+h}-x_n)}\right| + \frac{h}{N}\right).\end{aligned}$$

For $h \in \mathbb{Z}^+$, since $\langle x_{n+h} - x_n\rangle$ is equidistributed in $[0, 1)$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(x_{n+h}-x_n)} = 0.$$

Hence

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n}\right|^2 \leq \frac{C}{H}.$$

As H can be arbitrarily large, this implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0.$$

The result follows from Weyl's criterion. $\square$