MATH 3093 Fourier Analysis

Tutorial 12

1. Let $\mathcal{F}_R(t)$ be the Fejér kernel on the real line, that is

$$\mathcal{F}_R(t) = \begin{cases} R\left(\dfrac{\sin \pi t R}{\pi t R}\right)^2 & \text{if } t \neq 0, \\ R & \text{if } t = 0. \end{cases}$$

Let $F_N(x)$ be the Fejér kernel for 1-periodic functions, that is

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

Show that

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x).$$

2. Suppose f is a function of moderate decrease and that its Fourier transform $\hat{f}$ is supported in $I = [-1/2, 1/2]$.

(a) Prove the following reconstruction formula:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) K(x-n), \quad \text{where } K(y) = \frac{\sin \pi y}{\pi y}.$$

(**Hint:** Applying Poisson summation formula to $\hat{f}$ and observing that

$$\chi_I \sum_{n=-\infty}^{\infty} \hat{f}(\xi + n) = \hat{f}(\xi),$$

one obtains

$$\hat{f}(\xi) = \chi_I \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}.$$

The result follows by taking Fourier transform on both sides.)

(b) Let $\lambda > 1$. Show that

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right), \quad \text{where } K_\lambda(y) = \frac{\cos \pi y - \cos \pi \lambda y}{\pi^2(\lambda - 1) y^2}.$$

(**Hint:** Let $g(x) = f(x/\lambda)$. Then $\hat{g}(\xi) = \lambda \hat{f}(\lambda \xi)$ and $\hat{g}$ is supported in $[-1/(2\lambda), 1/(2\lambda)] \subset [-1/2, 1/2]$. Apply Poisson summation formula to $\hat{g}$ to conclude that

$$\sum_{n=-\infty}^{\infty} \hat{g}(\xi + n) = \sum_{n=-\infty}^{\infty} g(n) e^{-2\pi i n \xi}.$$

In particular, since g is supported in I,

$$\hat{g}(\xi) = \chi_I \sum_{n=-\infty}^{\infty} g(n) e^{-2\pi i n \xi}.$$

Let H be the function in Figure 2 on p.168. Denote $\tilde{H}(\xi) = H(\lambda\xi)$. Then

$$\hat{g} = \tilde{H} \cdot \hat{g} \quad \text{and} \quad \chi_I \cdot \tilde{H} = \tilde{H},$$

so that

$$\hat{g}(\xi) = \sum_{n=-\infty}^{\infty} g(n) \tilde{H}(\xi) e^{-2\pi i n \xi}.$$

Substitute $\xi' = \lambda\xi$, we have

$$\hat{f}(\xi') = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) H(\xi') e^{-2\pi i (n/\lambda)\xi'}.$$

The result then follows by taking Fourier transform on both sides. The Fourier transform of H can be computed by noting

$$H(x) = \frac{\lambda}{\lambda - 1}\phi\left(\frac{x}{\lambda/2}\right) - \frac{1}{\lambda - 1}\phi\left(\frac{x}{1/2}\right),$$

where $\phi(x) = \chi_{[-1,1]}(1 - |x|)$.)

(c) Show that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f(n)|^2.$$