Ex 14, 20 in Ch 5

Fourier transform, a fcn on $\mathbb{R}$

1. • Recall that if f, $\hat{f}$ are continuous of moderate decrease, then

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

(Poisson summation formula)

Also, one can show that

$$\sum_{n\in\mathbb{Z}} \hat{f}(x+n) = \sum_{n\in\mathbb{Z}} f(n) e^{-2\pi i n x}$$

by Fourier inversion formula.

• Also by Ex 2 in Ch. 5 :

if we put $g : \mathbb{R} \longrightarrow \mathbb{R}$

$$g(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases},$$

then

$$\hat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi} \right)^2$$

Hence, $\tilde{F}_R(t) = \hat{h}_R(t)$, where

$$h_R(x) = g\left(\frac{x}{R}\right)$$

$$\tilde{F}_R(t) = \begin{cases} R\left(\frac{\sin \pi t R}{\pi t R}\right)^2 & \text{if } t \neq 0 \\ R & \text{if } t = 0 \end{cases}$$

Now, by Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} \hat{h}_N(x+n) = \sum_{n=-\infty}^{\infty} h_N(n) e^{-2\pi i n x}$$

$$\text{LHS} = \sum_{n=-\infty}^{\infty} \tilde{F}_N(x+n)$$

RHS :

$$h_N(n) = \begin{cases} 1 - \frac{|n|}{N} & \text{if } |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\text{RHS} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{-2\pi i n x}$$

$$= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = F_N(x)$$

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2. (a) Using Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi} \quad \forall \xi \in \mathbb{R}$$

If $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, $\hat{f}(\xi+n) \neq 0$ only if $n=0$

(because we assume that $\operatorname{supp} \hat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]$)

Hence, $$\hat{f}(\xi) = \chi(\xi) \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}$$

where $$\chi(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$$

Fourier inversion formula

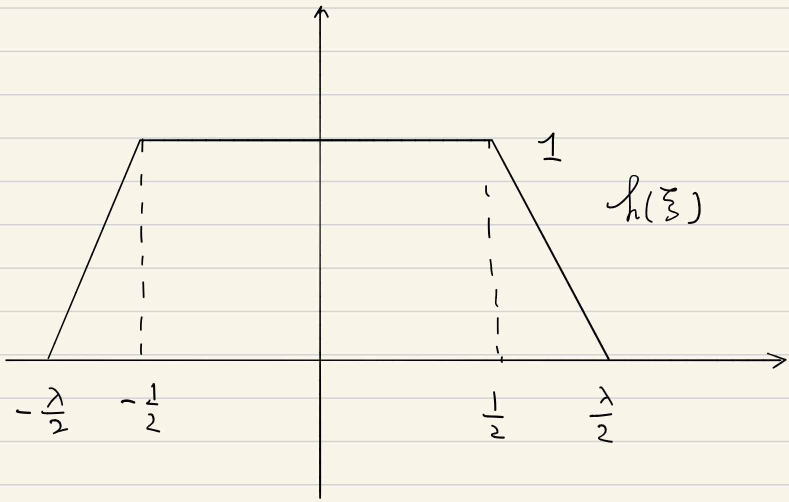
$$\Rightarrow \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(n) e^{2\pi i \xi (x-n)} \, d\xi$$

$$= \sum_{n=-\infty}^{\infty} f(n) \left(\frac{e^{i\pi(x-n)} - e^{-i\pi(x-n)}}{2\pi i (x-n)} \right)$$

$$= \sum_{n=-\infty}^{\infty} f(n) K(x-n)$$

(b) Hint in textbook Q20 in Ch.5 suggests us to consider the function below



instead of $\chi_{[-\frac{1}{2},\frac{1}{2}]}$

We normalize the interval $[-\frac{\lambda}{2}, \frac{\lambda}{2}]$

$$\begin{array}{ccc} [-\frac{1}{2}, \frac{1}{2}] & \subseteq & [-\frac{\lambda}{2}, \frac{\lambda}{2}] \\ \downarrow & & \downarrow \\ [-\frac{1}{2\lambda}, \frac{1}{2\lambda}] & \subseteq & [-\frac{1}{2}, \frac{1}{2}] \end{array}$$

We dilate the fcn f so that

$\text{supp}\,\hat{f}_\lambda \subseteq [-\frac{1}{2\lambda}, \frac{1}{2\lambda}]$

Let $f_\lambda(x) = f(\frac{x}{\lambda})$, then $\hat{f_\lambda}(\xi) = \lambda \hat{f}(\lambda\xi)$

Poisson summation formula

$$\Rightarrow \sum_{n\in\mathbb{Z}} \hat{f}_\lambda(\xi+n) = \sum_{n\in\mathbb{Z}} f_\lambda(n)\, e^{-2\pi i \xi n}$$

$$= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right) e^{-2\pi i \xi n} \qquad \forall \xi \in \mathbb{R}$$

For $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, we have

$$\hat{f}_\lambda(\xi) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right) e^{-2\pi i \xi n}$$

$$= \sum_{n\in\mathbb{Z}} h(\lambda\xi)\, f\left(\frac{n}{\lambda}\right) e^{-2\pi i \xi n}$$

Notice that

$$h(\xi) = \frac{\lambda}{\lambda-1}\,\phi\left(\frac{2}{\lambda}\xi\right) - \frac{1}{\lambda-1}\,\phi(2\xi)$$

where $\phi(\xi) = \begin{cases} 1 - |\xi| & \text{if } |\xi| \le 1 \\ 0 & \text{otherwise} \end{cases}$

It is known that $\hat{\phi}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$

Fourier inversion formula

$$\Rightarrow \quad f_\lambda(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right) \int_{-\infty}^{\infty} h(\lambda\xi)\, e^{-2\pi i\xi(n-x)}\, d\xi$$

$$= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right) \frac{1}{\lambda} \hat{h}\left(\frac{n-x}{\lambda}\right)$$

Note $\hat{h}(y)$

$$= \frac{\lambda}{\lambda-1}\cdot\frac{\lambda}{2}\left(\frac{\sin\frac{\pi\lambda y}{2}}{\frac{\pi\lambda y}{2}}\right)^2 - \frac{1}{2(\lambda-1)}\left(\frac{\sin\frac{\pi y}{2}}{\frac{\pi y}{2}}\right)^2$$

$$= \frac{2}{\lambda-1}\frac{1}{\pi^2y^2}\sin^2\left(\frac{\pi\lambda y}{2}\right) - \frac{2}{\lambda-1}\frac{1}{\pi^2y^2}\sin^2\left(\frac{\pi y}{2}\right)$$

$$= \frac{2}{\lambda-1}\frac{1}{\pi^2y^2}\left(\frac{1}{2}(1-\cos(\pi\lambda y)) - \frac{1}{2}(1-\cos\pi y)\right)$$

$$= \frac{1}{\pi^2(\lambda-1)}\frac{1}{y^2}\left(\cos\pi y - \cos\pi\lambda y\right)$$

$$= K_\lambda(y)$$

$$\therefore \quad f(x) = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(\frac{n}{\lambda} - x\right)$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right)$$

(c).

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi} \qquad \forall\ \xi \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

Parseval's identity

$$\Rightarrow \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\xi)|^2 \, d\xi = \sum_{n=-\infty}^{\infty} |f(n)|^2$$

$\because \operatorname{supp}\hat{f} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ $\qquad \|$

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

Plancherel formula.