MATH 3093 Fourier Analysis

Tutorial 11 (April 27)

The following were discussed in the tutorial this week:

1. Define
$$u(x,t) := \frac{x}{t}\mathcal{H}_t(x),$$
where $\mathcal{H}_t(x)$ is the heat kernel given by
$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}.$$
Show that
   (a) u satisfies the heat equation for $t > 0$,
   (b) $\lim_{t\to 0} u(x,t) = 0$ for every x,
   (c) u is *not* continuous at the origin.

2. Consider the following variant of the heat equation:
$$\begin{cases} x^2\dfrac{\partial^2 u}{\partial x^2} + ax\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t}, & \text{for } 0 < x < \infty, \quad t > 0. \\ u(x,0) = f(x), \end{cases} \tag{1}$$
Make the change of variables $x = e^{-y}$, $-\infty < y < \infty$. Set $U(y,t) = u(e^{-y},t)$ and $F(y) = f(e^{-y})$. Then equation (1) reduces to
$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} + (1-a)\dfrac{\partial U}{\partial y} = \dfrac{\partial U}{\partial t}, & \text{for } -\infty < y < \infty, \quad t > 0. \\ U(y,0) = F(y), \end{cases} \tag{2}$$
Take the Fourier transform in the variable y in (2) and show that
$$\hat{U}(\xi,t) = \hat{F}(\xi)e^{(-4\pi^2\xi^2 + (1-a)2\pi i\xi)t}.$$
Hence show that the solution of (1) is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-(\log(v/x)+(1-a)t)^2/(4t)}f(v)\frac{dv}{v}.$$

3. Recall that the Fourier transform of $h(x) := e^{-|x|}\cos x$ is given by
$$\hat{h}(\xi) = \frac{2(2\pi\xi)^2 + 4}{(2\pi\xi)^4 + 4}.$$
Hence compute the integral
$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}dx.$$