MATH 3093 Fourier Analysis

Tutorial 10 (April 20)

The following were discussed in the tutorial this week:

1. Recall that the Fourier transform of the Gaussian $\psi(x) := e^{-\pi x^2}$ is itself, that is $\mathcal{F}[\psi] = \psi$. We would like to generate other functions f in $\mathcal{S}(\mathbb{R})$ that, up to a constant multiple, are their own Fourier transform, that is

$$\mathcal{F}[f] = \lambda f \quad \text{for some scalar } \lambda.$$

Viewing $\mathcal{F}$ as a linear operator on $\mathcal{S}(\mathbb{R})$, this is to find the eigenfunctions of $\mathcal{F}$.

By the Fourier inversion formula, one can show that

$$\mathcal{F}[\mathcal{F}[f]](x) = f(-x).$$

Denote by $\mathcal{F}^n = \mathcal{F} \circ \cdots \circ \mathcal{F}$ the composition of n Fourier transforms. Define the operator $\iota : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $\iota[f](x) = f(-x)$. We have

$$\mathcal{F}^2 = \iota,$$

and hence

$$\mathcal{F}^4 = I,$$

where I is the identity operator. Then the eigenvalues λ of $\mathcal{F}$ satisfy $\lambda^4 = 1$, so that $\lambda = \pm 1, \pm i$, or $\lambda = (-1)^n$ for $n = 0, 1, 2, 3$. Consider the algebraic relation

$$x^4 - 1 = x^4 - \lambda^4 = (x - \lambda)(x^3 + \lambda x^2 + \lambda^2 x + \lambda^3).$$

Since $\mathcal{F}^4 = I$, we have

$$0 = \mathcal{F}^4 - I = (\mathcal{F} - \lambda I)(\mathcal{F}^3 + \lambda \mathcal{F}^2 + \lambda^2 \mathcal{F} + \lambda^3 I).$$

If we multiply λ on both sides, it becomes

$$0 = (\mathcal{F} - \lambda I)(I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3)$$

Therefore, for any $f \in \mathcal{S}(\mathbb{R})$, if we put

$$f_\lambda := \frac{1}{4}\left(I + \lambda^3 \mathcal{F} + \lambda^2 \mathcal{F}^2 + \lambda \mathcal{F}^3\right)[f],$$

then it satisfies

$$(\mathcal{F} - \lambda I)[f_\lambda] = 0.$$

Moreover, f admits the following decomposition

$$f = f_{+1} + f_{-i} + f_{-1} + f_{-i}.$$

If f_λ is non-zero, then it is an eigenfunction of $\mathcal{F}$ corresponding to λ.

Consider $g(x) := (2\pi x)^n e^{-\pi x^2} = (i)^n(-2\pi i x)^n \psi(x)$. Then the Fourier transform of g is given by

$$\begin{aligned}\mathcal{F}[g](x) &= (i)^n \psi^{(n)}(x) \\ &= (-i)^n q_n(x)\psi(x),\end{aligned}$$

where $q_n(x)$ is a degree n polynomial of the form $(2\pi x)^n +$lower degree terms. Thus it is natural to guess that for each n, there is a polynomial $p_n(x)$ of degree n such that

$$\mathcal{F}[p_n(x)e^{-\pi x^2}] = (-i)^n p_n(x)e^{-\pi x^2}.$$

If $g_{(-i)^n}$ is a non-zero function, such $p_n(x)$ can be found. Compute that

$$\mathcal{F}^2[g](x) = g(-x) = (-1)^n(2\pi x)^n \psi(x),$$

and

$$\mathcal{F}^3[g](x) = \mathcal{F}[g](-x) = (i)^n \psi^{(n)}(-x) = (-i)^n \psi^{(n)}(x),$$

here we use $\mathcal{F}^2 = \iota$, $\psi(x)$ is even and the fact that the derivative of an even function is odd and vice versa. Therefore

$$\begin{aligned}g_{(-i)^n}(x) &= \frac{1}{4}\left(g + (-i)^{3n}\mathcal{F}[g] + (-i)^{2n}\mathcal{F}^2[g] + (-i)^n\mathcal{F}^3[g]\right)(x) \\ &= \frac{1}{2}\left((2\pi x)^n\psi(x) + (-1)^n\psi^{(n)}(x)\right) \\ &= [(2\pi x)^n + Q(x)]\psi(x),\end{aligned}$$

where $Q(x)$ is a polynomial of degree $\le n-1$. Let $h_n := g_{(-i)^n}$. Then $\{h_n\}$ is a sequence of eigenfunctions of $\mathcal{F}$ that takes the form $\{p_n(x)e^{-\pi x^2}\}$ for some polynomials p_n of degree n. Listed below are the first few terms of the sequence $\{h_n\}$:

$$h_0(x) = e^{-\pi x^2}, \quad h_1(x) = 2\pi x e^{-\pi x^2}, \quad h_2(x) = (4\pi^2x^2 - \pi)e^{-\pi x^2}.$$

2. Suppose f is a continuous function of moderate decrease such that

$$\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}dy = 0 \quad \text{for all } x \in \mathbb{R}.$$

Show that $f = 0$.

[Hint: Consider $f * e^{-x^2}$.]

Solution. Put $g(z) = e^{-z^2}$ and notice that

$$\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\, dy = e^{x^2}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2}\, dy = f * g(x)\cdot e^{x^2}$$

By assumption, we have $f * g(x) = 0$ for all $x \in \mathbb{R}$. By section 1.7 on p.144, we see that $\hat{f}(\xi)\hat{g})(\xi) = 0$ for all $\xi \in \mathbb{R}$. Moreover, by proposition 1.2 and theorem 1.4,

$$\hat{g}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2}$$

This shows that $\hat{f} = 0$. Note that both f and $\hat{f}$ is of moderate decrease, Fourier inversion formula shows that f can be recovered from $\hat{f}$, which is 0. ◀