MATH 3093 Fourier Analysis

Tutorial 4 (February 24)

The following will be discussed in the tutorial this week:

1. Let D_N denote the Dirichlet kernel

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)| d\theta.$$

(a) Show that

$$L_N \geq c \log N$$

for some constant $c > 0$.

(b) Show that, for each $n \geq 1$, there is a continuous function f_n on $[-\pi, \pi]$ such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c' \log n$.

Solution. (a) Recall that $|\sin x| \leq |x|$ for every $x \in \mathbb{R}$, we have

$$|D_N(\theta)| \geq \left|\frac{\sin((N+1/2)\theta)}{\theta/2}\right|$$

For the integral L_N, we first substitute $(N+1/2)\theta = x$, so that

$$\begin{aligned}
L_N &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin((N+1/2)\theta)}{\theta/2}\right| \mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{|\sin((N+1/2)\theta)|}{\theta/2} \mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{(N+1/2)\pi} \frac{|\sin x|}{\frac{x}{2(N+1/2)}} \frac{\mathrm{d}x}{N+1/2} \\
&= \frac{2}{\pi}\int_0^{(N+1/2)\pi} \frac{|\sin x|}{x} \mathrm{d}x \\
&\geq \frac{2}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} \mathrm{d}x \\
&= \frac{2}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{|\sin(k-1)\pi + y|}{(k-1)\pi + y} \mathrm{d}y \\
&= \frac{2}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{\sin y}{(k-1)\pi + y} \mathrm{d}y \\
&\geq \frac{2}{\pi}\sum_{k=1}^{N}\frac{1}{k\pi}\int_0^{\pi} \sin y \, \mathrm{d}y = \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}
\end{aligned}$$

By the comparison $\frac{1}{k} \geq \int_k^{k+1} \frac{1}{x}\,\mathrm{d}x$, we have $L_N \geq \frac{4}{\pi^2}\log(N+1)$.

(b) For each $n \geq 1$, we let

$$g_n(x) = \begin{cases} 1, & \text{if } D_n(x) \geq 0; \\ -1, & \text{if } D_x(x) < 0. \end{cases}$$

Notice that

$$S_n g_n(0) = g_n * D_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(x) D_n(-x) \mathrm{d}x.$$

Since $D_n(x)$ is an even function, we have

$$S_n g_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)| \mathrm{d}x = L_n.$$

The only problem is that g_n is not a continuous function.
For the fixed number n, by Lemma 3.2 in Chapter 2, there is a sequence of continuous functions $\{f_k\}$ on the circle $[-\pi, \pi]$, approximating g_n in the sense below.

(i) $\|f_k\|_\infty \leq \|g_n\|_\infty = 1$
(ii) $\int_{-\pi}^{\pi} |f_k(x) - g_n(x)| \,\mathrm{d}x \to 0$ as $k \to \infty$.

Now,

$$|S_n f_k(0) - S_n g_n(0)| = |(f_k - g_n) * D_n(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_k(x) - g_n(x)| M_n \,\mathrm{d}x,$$

where M_n is the bound for the continuous function D_n on $[-\pi, \pi]$.
By (ii), $S_n f_k(0) - S_n g_n(0) \to 0$ as $k \to \infty$. So for some large k, f_k will be the desired function. (We may put $c' = \frac{c}{2}$).

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2. Suppose f is a Riemann integrable function on $[-\pi, \pi]$ such that $|\hat{f}(n)| \leq K/|n|$. Show that $\|S_N(f)\|_\infty \leq \|f\|_\infty + 4K$ on $[-\pi, \pi]$.

Solution. Recall that, for any $r \in (0, 1)$ and $x \in [-\pi, \pi]$,

$$A_r(f)(x) := \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{inx} = f * P_r(x),$$

where $P_r(x)$ is the Poisson kernel given by

$$P_r(x) = \frac{1 - r^2}{1 - 2r \cos x + r^2} \geq 0.$$

Note that for any $r \in (0, 1), x \in [-\pi, \pi]$,

$$|A_r(f)(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x - y)| P_r(y) dy \leq \|f\|_\infty \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(y) dy = \|f\|_\infty.$$

Moreover, we have

$$\begin{aligned} |S_N(f)(x) - A_r(f)(x)| &\le K \sum_{0<|n|\le N} \frac{1-r^{|n|}}{n} + \frac{K}{N} \sum_{|n|\ge N+1} r^{|n|} \\ &\le 2KN(1-r) + \frac{2K}{N(1-r)} \\ &= 4K, \end{aligned}$$

if we take $r = 1 - 1/N$. The result then follows. ◀

3. Suppose a series $\sum_{k=1}^{\infty} c_k$ is Cesàro summable to s.
   If $c_k = O(1/k)$ (i.e. $|kc_k| \le M$ for all k), show that $\sum_{k=1}^{\infty} c_k$ converges to s.

   Solution. Note that

$$\sigma_n = \tfrac{1}{n} \textstyle\sum_{k=1}^{n} s_k = \tfrac{1}{n} \sum_{k=1}^{n} (n-k+1)\, c_k$$

$$s_n = \textstyle\sum_{k=1}^{n} c_k$$

For $m > n$, we have

$$\begin{aligned} m\sigma_m - n\sigma_n &= \sum_{k=1}^{m} (m-k+1)\, c_k - \sum_{k=1}^{n} (n-k+1)\, c_k \\ &= \sum_{k=1}^{n} (m-n) c_k + \sum_{k=n+1}^{m} (m-k+1) c_k \\ &= (m-n) s_n + \sum_{k=n}^{m-1} (m-k) c_{k+1} \end{aligned}$$

From which we deduce that for $m > n$, we have

$$s_n - \sigma_n = \frac{m}{m-n}(\sigma_m - \sigma_n) - \frac{m}{m-n} \sum_{k=n}^{m-1} (1 - \frac{k}{m}) c_{k+1}$$

Now we claim that $s_n - \sigma_n$ goes to 0 as n goes to ∞. For each $\epsilon > 0$ and $n \in \mathbb{N}$, we consider particularly $m = n + [\epsilon n]$, where $[\epsilon n]$ is the largest integer less that ϵn.

It can be seen that $\lim_{n\to\infty} \frac{m}{m-n} = \frac{1+\epsilon}{\epsilon}$, hence $\frac{m}{m-n} < \frac{1+\epsilon}{\epsilon} + 1$ for some large n, say $n \ge N_1$. On the other hand, by the convergence of $\lim_{n\to\infty} \sigma_n$, $|\sigma_m - \sigma_n| < \epsilon^2$ for $n \ge N_2$.

Therefore, the first term can be controled in the sense that

(we assumed $\epsilon < \frac{1}{2}$ for simplicity.)

$$\left| \frac{m}{m-n}(\sigma_m - \sigma_n) \right| \le \epsilon(2\epsilon + 1) \le 2\epsilon \quad \text{for } n \ge \max\{N_1, N_2\}.$$

For the second term

$$\frac{m}{m-n}\sum_{k=n}^{m-1}(1-\frac{k}{m})c_{k+1}=\frac{m}{m-n}\sum_{k=n}^{m-1}(\frac{1}{k}-\frac{1}{m})kc_{k+1},$$

$|kc_{k+1}|=|(k+1)c_{k+1}-c_{k+1}|\leq M+\frac{M}{k+1}\leq 2M$. By comparison $\frac{1}{k}\leq\int_{k-1}^{k}\frac{1}{x}\,\mathrm{d}x$, we have

$$\sum_{k=n}^{m-1}\frac{1}{k}\leq\ln\left(\frac{m-1}{n-1}\right)=\ln(1+\frac{[\epsilon n]}{n-1})\leq\frac{\epsilon n}{n-1}$$

Therefore,

$$\frac{m}{m-n}\sum_{k=n}^{m-1}(1-\frac{k}{m})c_{k+1}\leq\frac{m}{m-n}(\frac{\epsilon n}{n-1}-\frac{m-n}{m})2M$$

Note $\frac{n}{n-1}\leq 1+\epsilon$ for large n, say $n\geq N_3$, and recall that $\frac{m}{m-n}<\frac{1+\epsilon}{\epsilon}+1$ for $n\geq N_1$. The second term is now bounded by $2M((1+2\epsilon)(1+\epsilon)-1)\leq 10M\epsilon$, when ϵ is small enough and $n\geq\max\{N_1,N_3\}$. This completes the proof.

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