MATH 3093 Fourier Analysis

Tutorial 5 (March 2)

The following will be discussed in the tutorial this week:

Please refer to exercises 15,16 in Chapter 3 of our textbook, by E.M. Stein and R. Shakarchi.

Let f be a 2π-periodic Riemann integrable function on $[-\pi, \pi]$, and $0 < \alpha \leq 1$.

1. (Hölder Continuity)

   (a) f is said to be α-Hölder continuous at x_0 if there is a constant $C = C(x_0) > 0$ such that
   $$|f(x_0 + h) - f(x_0)| \leq C|h|^\alpha \quad \forall h.$$

   (b) f is said to be α-Hölder continuous if there is a constant $C > 0$ such that
   $$|f(x_0 + h) - f(x_0)| \leq C|h|^\alpha \quad \forall x, h.$$

2. It is proved in lecture that if f is α-Hölder continuous at x_0, then
   $$S_N f(x_0) \to f(x_0) \quad \text{as } N \to \infty.$$
   If f is α-Hölder continuous, then we can say more, i.e.
   $$S_N f \to f \text{ uniformly} \quad \text{as } N \to \infty.$$
   While the general case will not be proved in the tutorial class, for $1/2 < \alpha$, it is a consequence of item 5. (Ex 16)

3. If f is α-Hölder continuous, can we say something about the order of decay of $\hat{f}(n)$?

   Yes! We have $\hat{f}(n) = O(1/|n|^\alpha)$. (Ex. 15)

4. The order of decay cannot be improved in general. As an example, one can show that, for $0 < \alpha < 1$, (Ex. 15)
   $$f(x) := \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}$$
   is α-Hölder continuous and satisfies $\hat{f}(n) = 1/n^\alpha$ for $n = 2^k$, $k = 0, 1, \ldots$

5. If f is α-Hölder continuous with $1/2 < \alpha \leq 1$, we can obtain a stronger convergence for the Fourier series of f, namely the absolute convergence, that is (Ex. 16)
   $$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$