TA's solution to 3093 assignment 7

Ch4, Ex11. (2 marks)

We have

$$\int_0^1 |f * H_t(x) - f(x)|^2\,dx = \sum_{-\infty}^{\infty} \left|\widehat{f * H_t}(n) - \widehat{f}(n)\right|^2$$
$$= \sum_{-\infty}^{\infty} \left|\widehat{f}(n)\widehat{H_t}(n) - \widehat{f}(n)\right|^2 = \sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 \left(1 - e^{-4\pi^2 n^2 t}\right)^2.$$

Since the series $\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2$ is convergent, it follows from Weierstrass M-Test that the function $F(t) := \sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 \left(1 - e^{-4\pi^2 n^2 t}\right)^2$ is uniformly convergent on $[0, \infty)$. Therefore F is continuous on $[0,\infty)$ and so $\lim_{t\downarrow 0} F(t) = F(0) = 0$. Done.

Ex12. (2 marks)*

We are given that for some 1-periodic Riemann integrable function $f(x) \sim \sum_{-\infty}^{\infty} a_n e^{2in\pi x}$, the function $u : [0,1] \times [0,\infty) \to \mathbb{C}$ defined by

$$u(x,t) := \begin{cases} \sum_{-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2in\pi x} & \text{if } t > 0 \\ f(x) & \text{if } t = 0 \end{cases}$$

satisfies

(a) $u_t = u_{xx}$ on $[0,1] \times (0,\infty)$;

(b) For $t > 0$, we have $u(x,t) = (f * H_t)(x)$, where $H_t(x) := \sum_{-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2in\pi x}$ and "$*$" is defined by

$$(g * h)(x) := \int_0^1 g(x-y)h(y)dy$$

for any 1-periodic Riemann integrable functions g, h;

(c) $\lim_{t\downarrow 0} \int_0^1 |u(x,t) - f(x)|^2\,dx = 0$.

Let $F : [0,2\pi] \to \mathbb{C}$, $U : [0,2\pi] \times [0,\infty) \to \mathbb{C}$, and (for $\tau > 0$) $h_\tau : [0,2\pi] \to \mathbb{C}$ be defined by

$$F(\theta) := f(\frac{\theta}{2\pi}), \quad U(\theta,\tau) := u(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2}), \quad h_\tau(\theta) := H_{\tau/(4\pi^2)}(\frac{\theta}{2\pi}) = \sum_{-\infty}^{\infty} e^{-n^2\tau} e^{in\theta}.$$

Then we have the following results:

(i) By the definition of U, we have

$$U_\tau(\theta,\tau) = \frac{u_t(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2})}{4\pi^2}, \quad U_\theta(\theta,\tau) = \frac{u_x(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2})}{2\pi}, \quad U_{\theta\theta}(\theta,\tau) = \frac{u_{xx}(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2})}{4\pi^2}.$$

Hence by (a), we have $U_\tau = U_{\theta\theta}$ on $[0,2\pi] \times (0,\infty)$.

*I find this question quite ambiguous and I may misunderstand it.

(ii) For $\tau > 0$, by the definition of U and u, we have

$$U(\theta,\tau) = u(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2}) = \sum_{-\infty}^{\infty} a_n e^{-4\pi^2 n^2 \frac{\tau}{4\pi^2}} e^{2in\pi\frac{\theta}{2\pi}} = \sum_{-\infty}^{\infty} a_n e^{-n^2\tau} e^{in\theta},$$

and by (b) we have

$$\begin{aligned} U(\theta,\tau) &= u(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2}) = (f * H_{\tau/(4\pi^2)})(\frac{\theta}{2\pi}) = \int_0^1 f(\frac{\theta}{2\pi} - y) H_{\tau/(4\pi^2)}(y)dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\frac{\theta}{2\pi} - \frac{\xi}{2\pi}) H_{\tau/(4\pi^2)}(\frac{\xi}{2\pi})d\xi = \frac{1}{2\pi}\int_0^{2\pi} F(\theta-\xi)h_\tau(\xi)d\xi := F \star h_\tau, \end{aligned}$$

where "$\star$" is defined by

$$(G \star H)(\theta) := \frac{1}{2\pi}\int_0^{2\pi} G(\theta-\xi)H(\xi)d\xi$$

for any 2π-periodic Riemann integrable functions G, H.

(iii) By the definition of U and u, we have $U(\theta, 0) = u(\frac{\theta}{2\pi}, 0) = f(\frac{\theta}{2\pi}) = F(\theta)$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} |U(\theta,\tau) - F(\theta)|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left| u(\frac{\theta}{2\pi}, \frac{\tau}{4\pi^2}) - f(\frac{\theta}{2\pi})\right|^2 d\theta = \int_0^1 \left| u(x, \frac{\tau}{4\pi^2}) - f(x)\right|^2 dx,$$

by (c) we have

$$\lim_{\tau\downarrow 0} \frac{1}{2\pi}\int_0^{2\pi} |U(\theta,\tau) - F(\theta)|^2 d\theta = 0.$$

Similarly, whenever $x_0 \in [0,1]$ satisfies $\lim_{t\downarrow 0} u(x_0, t) = f(x_0)$, letting $\theta_0 := 2\pi x_0$ we have

$$\lim_{\tau\downarrow 0} U(\theta_0,\tau) = \lim_{\tau\downarrow 0} u(\frac{\theta_0}{2\pi}, \frac{\tau}{4\pi^2}) = f(\frac{\theta_0}{2\pi}) = F(\theta_0).$$

Done.

Ch5, Ex1. (6 marks) (It seems that you already have good solution from tutorial??)

We remark that when doing part (c), in order to use the result of part (b), we have to check that $F(y) := \widehat{f}(y)e^{2i\pi yx}$ is continuous and of moderate decrease. To show F is continuous, we want to show that $\widehat{f}$ is continuous. It may be done by using that f is continuous with compact support $[-M, M]$. Here is another approach: by hypothesis, $\widehat{f}$ is of moderate decrease. Therefore, by the exact definition of "moderate decrease" in the textbook, $\widehat{f}$ is continuous[†].

[†]This idea is suggested by a student. Sorry that I previously misunderstood that the "moderate decrease" condition was about magnitude only and not about smoothness. If I marked your work wrongly, please feel free to contact me.