TA's solution* to 3093 assignment 2

Ch2, Ex4. (4 marks)

(a) Note that $f(\theta) = \begin{cases} \theta(\pi+\theta) & \text{if } \theta \in [-\pi, 0] \\ \theta(\pi-\theta) & \text{if } \theta \in [0, \pi]. \end{cases}$

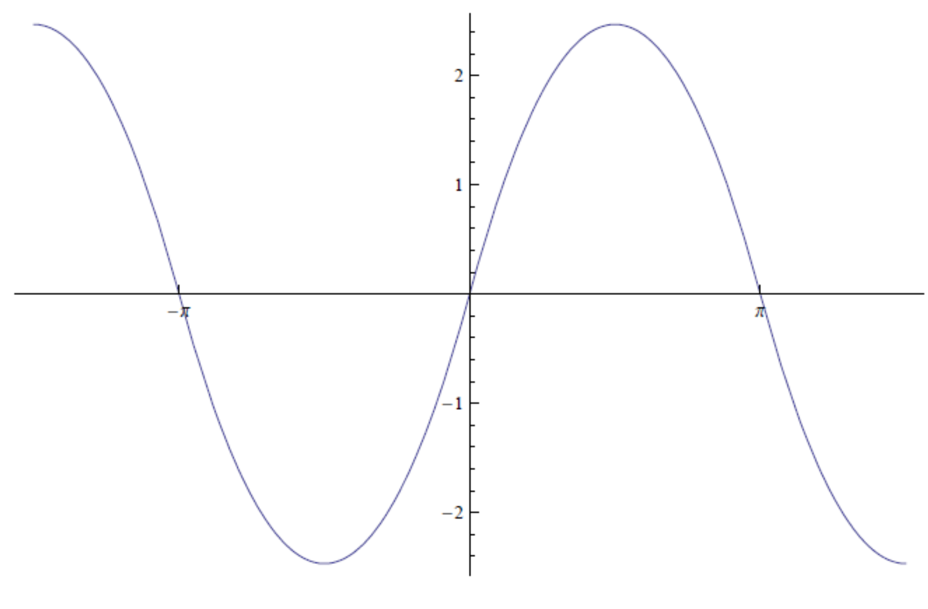

FIGURE 1. $f(\theta) = \theta(\pi - \theta)$, with odd extension

(b) We have $\widehat{f}(0) = 0$. For $n \neq 0$, we calculate the Fourier coefficients as follows:

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-i\sin n\theta)d\theta \qquad (\because f(\theta)\cos n\theta \text{ is odd in } [-\pi,\pi]) \\ &= \frac{-i}{\pi}\int_0^{\pi} \theta(\pi-\theta)\sin n\theta \, d\theta. \qquad (\because f(\theta)\sin n\theta \text{ is even in } [-\pi,\pi]) \end{aligned}$$

Using integration by parts and $\cos n\pi = (-1)^n$, we have

$$\int_0^{\pi} \theta \sin n\theta \, d\theta = \frac{-1}{n}\left[\pi(-1)^n - \int_0^{\pi}\cos n\theta \, d\theta\right] = \frac{-\pi(-1)^n}{n},$$

$$\begin{aligned} \int_0^{\pi} \theta^2 \sin n\theta \, d\theta &= \frac{-1}{n}\left[\pi^2(-1)^n - 2\int_0^{\pi}\theta\cos n\theta \, d\theta\right] = \frac{-\pi^2(-1)^n}{n} + \frac{2}{n^2}\left[-\int_0^{\pi}\sin n\theta \, d\theta\right] \\ &= \frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}[(-1)^n - 1]. \end{aligned}$$

As a result

$$\widehat{f}(n) = \frac{-i}{\pi}\cdot\frac{-2}{n^3}[(-1)^n - 1] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4i}{\pi n^3} & \text{if } n \text{ is odd.} \end{cases}$$

*This solution is adapted from the work by former TAs.

This shows the Fourier series of f is given by

$$\sum_{\substack{n\in\mathbb{Z}\\ n \text{ odd}}} \frac{-4i}{\pi n^3} e^{in\theta} = \sum_{\substack{n\in\mathbb{Z}\\ n \text{ odd}}} \frac{-4i}{\pi n^3} i\sin n\theta = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{8}{\pi n^3}\sin n\theta.$$

Since $\sum \left|\widehat{f}(n)\right| \leq C\sum \frac{1}{n^3} < \infty$ for some constant $C > 0$, the Fourier series is equal to $f^\dagger$.

Another approach for the integration:

We have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta &= \frac{1}{2\pi}\int_0^{\pi} \theta(\pi-\theta)e^{-in\theta}\ d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} \theta(\pi+\theta)e^{-in\theta}\ d\theta \\
&= \frac{1}{2\pi}\int_0^{\pi} \theta(\pi-\theta)e^{-in\theta}\ d\theta + \frac{1}{2\pi}\int_0^{\pi} (t-\pi)(\pi+(t-\pi))e^{-in(t-\pi)}\ dt \\
&= \frac{[1-e^{in\pi}]}{2\pi}\int_0^{\pi} \theta(\pi-\theta)e^{-in\theta}\ d\theta \\
&= \frac{[1-e^{in\pi}]}{2\pi}\int_{-\pi/2}^{\pi/2} (\frac{\pi}{2}-v)(\frac{\pi}{2}+v)e^{-in(\frac{\pi}{2}-v)}\ dv \\
&= \frac{-i\sin\frac{n\pi}{2}}{\pi}\int_{-\pi/2}^{\pi/2} (\frac{\pi^2}{4}-v^2)e^{inv}\ dv.
\end{aligned}$$

By thinking of integration by parts, an anti-derivative of the integrand above is of the form $(Av^2+Bv+C)e^{inv}$ for some $A, B, C \in \mathbb{R}$. Hence the above is

$$\begin{aligned}
&= \frac{-i\sin\frac{n\pi}{2}}{\pi}\left[(Av^2+Bv+C)e^{inv}\right]_{v=-\pi/2}^{v=\pi/2} \\
&= \frac{-i\sin\frac{n\pi}{2}}{\pi}\left[(A\frac{\pi^2}{4}+C)2i\sin\frac{n\pi}{2} + B\frac{\pi}{2}2\cos\frac{n\pi}{2}\right] \\
&= \frac{-i\sin\frac{n\pi}{2}}{\pi}\left[(A\frac{\pi^2}{4}+C)2i\sin\frac{n\pi}{2}\right] \qquad (\because 2\sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \sin n\pi = 0) \\
&= \frac{2\sin^2\frac{n\pi}{2}}{\pi}\left[A\frac{\pi^2}{4}+C\right].
\end{aligned}$$

By the definition of anti-derivative, we have

$$in(Av^2+Bv+C)e^{inv} + (2Av+B)e^{inv} = (\frac{\pi^2}{4}-v^2)e^{inv},$$

so by comparing the coefficients

$$\begin{cases} inA = -1, \\ inB + 2A = 0, \\ inC + B = \frac{\pi^2}{4}, \end{cases}$$

whence

$$A = \frac{-1}{in}, \quad inC + (\frac{-2}{n^2}) = \frac{\pi^2}{4} \Rightarrow C = \frac{\pi^2}{4in} + \frac{2}{in^3},$$

and therefore $\left[A\frac{\pi^2}{4}+C\right] = \frac{2}{in^3}$. The result follows.

†It is textbook Ch2 Corollary 2.3

Ex6. (4 marks)

(a)

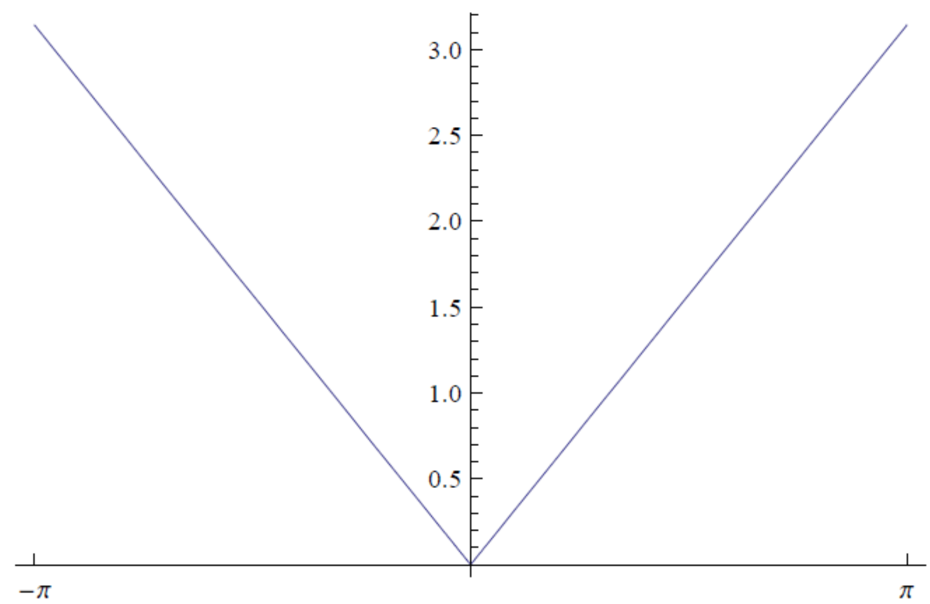

FIGURE 2. $f(\theta) = |\theta|$

(b) If $n = 0$, then $\widehat{f}(0) = \frac{1}{\pi}\int_0^\pi \theta d\theta = \frac{\pi}{2}$. Else if $n \neq 0$, then using f is even we have

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| \cos n\theta d\theta \\
&= \frac{1}{\pi}\int_0^\pi \theta \cos n\theta \ d\theta = \frac{1}{n\pi}\left(-\int_0^\pi \sin n\theta \ d\theta\right) \\
&= \frac{(-1)^n - 1}{n^2\pi}.
\end{aligned}$$

(c) By the result of part b,

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta} = \frac{\pi}{2} + \sum_{\substack{n\in\mathbb{Z} \\ n \text{ odd}}} \frac{-2}{n^2\pi}e^{in\theta} = \frac{\pi}{2} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{-4}{n^2\pi}\cos n\theta.$$

6(d). As $\sum |\widehat{f}(n)| \leq C \sum_n \frac{1}{n^2} < \infty$, for some constant $C > 0$, the Fourier series is equal to f (Corollary 2.3 of the book).

$$f(\theta) = \frac{\pi}{2} + \sum_{n\geq 1, n=odd} \frac{-4}{\pi n^2} \cos n\theta.$$

Taking $\theta = 0$, we have

$$0 = f(0) = \frac{\pi}{2} - \sum_{n\geq 1, n=odd} \frac{4}{\pi n^2}.$$

This implies that

$$\sum_{n\geq 1, n=odd} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n\geq 1, n=odd} \frac{1}{n^2} + \sum_{n\geq 1, n=even} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Ex10. (2 marks)

Since f is 2π-periodic, $f^{(i)}$ is 2π-periodic too for any $1 \leq i \leq k$.[‡] Consequently, $f^{(i)}(-\pi)e^{in\pi} = f^{(i)}(\pi)e^{-in\pi}$. Therefore, by successive integration by parts (for $n \neq 0$),

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{(in)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} f''(\theta)e^{-in\theta}d\theta \\
&= \cdots \\
&= \frac{1}{(in)^k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(\theta)e^{-in\theta}d\theta
\end{aligned}$$

As $f \in C^k$, so by the definition of C^k we have $f^{(k)}$ is continuous on $\mathbb{T}$. This means there exists $M > 0$ such that $\left|f^{(k)}(\theta)\right| < M$ for all θ. Hence

$$\left|\widehat{f}(n)\right| \leq \frac{1}{|n|^k}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f^{(k)}(\theta)\right| d\theta \leq \frac{M}{|n|^k}.$$

[‡]For example, $f'(x+2\pi) = \lim_{h\to 0} \frac{f(x+2\pi+h) - f(x+2\pi)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$.