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HÖLDER STABLE MINIMIZERS, TILT STABILITY, AND HÖLDER METRIC REGULARITY OF SUBDIFFERENTIALS*

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Abstract. Using techniques of variational analysis and dual techniques for smooth conjugate functions, for a local minimizer of a proper lower semicontinuous function f on a Banach space, $p \in (0,\ +\infty)$ and $q = \frac{1+p}{p}$, we prove that the following two properties are always equivalent: (i) $\bar{x}$ is a stable q-order minimizer of f and (ii) $\bar{x}$ is a tilt-stable p-order minimizer of f. We also consider their relationships in conjunction with the p-order strong metric regularity of the subdifferential mapping ∂f.

Key words. Hölder stable minimizer, Hölder tilt-stable minimizer, Hölder metric regularity, subdifferential, conjugate function

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1. Introduction.

For a proper lower semicontinuous function f on a Banach space X, recall (cf. [6, 10]) that $\bar{x} \in \mathrm{dom}(f)$ is a sharp minimizer of f if there exist positive constants κ and δ such that

$$\kappa\|x - \bar{x}\| \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta), \tag{1.1}$$

where $B_X(\bar{x}, \delta)$ denotes the open ball of X with center $\bar{x}$ and radius δ (and $B_X[\bar{x}, \delta]$ will be used to denote the corresponding closed ball). Clearly, (1.1) implies that $\arg\min_{x \in B_X(\bar{x},\delta)} f = \{\bar{x}\}$ is a singleton. Perhaps because of this, some authors (see [4]) use such names as "strong isolated local minimizer" or "strong local minimizer" instead of sharp minimizer. In the case when $\bar{x}$ is not a unique minimizer of f over $B_X(\bar{x}, \delta)$, Ferris [10] introduced the weak sharp minimizer notion: $\bar{x} \in \mathrm{dom}(f)$ is called a weak sharp minimizer of f if there exist $\kappa, r, \delta \in (0,\ +\infty)$ such that $f(\bar{x}) = \inf_{u \in B_X(\bar{x},r)} f(u)$ and

$$\kappa d(x, S(f, \bar{x}, r)) \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta), \tag{1.2}$$

where $S(f, \bar{x}, r) := \{x \in B_X(\bar{x}, r) :\ f(x) = \inf_{u \in B_X(\bar{x},r)} f(u)\}$. The notion of sharp/weak sharp minimizer has been recognized to be important in mathematical programming and has been well studied (see [6, 10, 29, 31, 34, 35] and references therein). However, both sharp minimizer and weak sharp minimizer are rather restrictive: for example, it can be shown easily that a smooth function has no sharp minimizers and that a smooth function also has no weak sharp minimizers (unless it is locally constant

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around some point). If $\|x-\bar{x}\|$ in (1.1) is replaced by $\|x-\bar{x}\|^q$ with some constant $q \geq 1$, namely, if there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa\|x-\bar{x}\|^q \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta), \tag{1.3}$$

then $\bar{x}$ is called a q-order sharp (local) minimizer of f (with modulus κ). Similarly, we define $\bar{x}$ to be a q-order weak sharp minimizer of f if there exist $\kappa, r, \delta \in (0, +\infty)$ such that $f(\bar{x}) = \inf_{u \in B_X(\bar{x}, r)} f(u)$ and

$$\kappa d(x, S(f, \bar{x}, r))^q \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta). \tag{1.4}$$

Clearly, if $1 < q$ and $\delta < 1$, (1.3) is weaker than (1.1). It has been found that the weak sharp minimum is closely related to the error bound in optimization, a notion that has received much attention (cf. [25, 26, 31, 32] and references therein). Recently, some authors considered Hölder error bounds (cf. [12, 30]). From the point of view of theoretical interest and applications, we address a natural question on how sensitive (1.3) or (1.4) is in responding to small perturbations to the function f. Noting that several authors considered small linear perturbations to f, we are concerned with the following notion of stable q-order strong minimizers, which has been studied under the name of strong stable local minimizer or uniform quadratic growth condition in the case when $q = 2$ (see [4, 8, 9, 20, 21, 22] and references therein).

Definition 1.1. *Let $\bar{x} \in \text{dom}(f)$ and $q \in [1, +\infty)$. We say that $\bar{x}$ is a stable q-order sharp minimizer of f if there exist $\delta, r, \kappa \in (0, +\infty)$ such that for each $u^* \in B_{X^*}(0, \delta)$ there exists $x_{u^*} \in B_X(\bar{x}, r)$, with $x_0 = \bar{x}$, satisfying the following property:*

$$\kappa\|x - x_{u^*}\|^q \leq f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X(\bar{x}, r),$$

where

$$f_{u^*} := f - u^*. \tag{1.5}$$

Motivated by the terminology of Poliquin and Rockafellar (see their seminal paper [28], where they initiated the tilt stability study for the case when X is finite dimensional and $p = 1$), let us introduce the following definition.

Definition 1.2. *Let $p \in (0, +\infty)$. We say that $\bar{x} \in \text{dom}(f)$ is a tilt-stable p-order (local) minimizer of f if there exist $r, \delta, L \in (0, \infty)$ and $M : B_{X^*}(0, \delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that*

$$f_{u^*}(M(u^*)) = \min_{x \in B_X[\bar{x}, r]} f_{u^*}(x) \quad \forall u^* \in B_{X^*}(0, \delta) \tag{1.6}$$

(where f_{u^*} is as in (1.5)) and

$$\|M(x^*) - M(u^*)\| \leq L\|x^* - u^*\|^p \quad \forall x^*, u^* \in B_{X^*}(0, \delta). \tag{1.7}$$

In this case we also say that $\bar{x}$ gives a tilt-stable p-order (local) minimum of f with modulus L.

To the best of our knowledge, the Hölder stability in the sense of Definition 1.1 or 1.2 has not been studied in the existing literature. Mordukhovich and Nghia [22] considered the so-called full Lipschitz and Hölder stability for the parameter case ($f(x)$ being replaced by $f(x, \nu)$); they studied the tilt stability of order one ($p = 1$) together with a parameter and the Hölder aspect (of order $\frac{1}{2}$) refers only to the parameter. In the special case when $q = 2$ and $p = 1$, many authors studied the stable minimizer and the tilt stability properties (cf. [1, 4, 8, 9, 17, 20, 21, 23, 24, 28]). In particular, the following interesting result has been established (cf. [28, 8, 20]).

THEOREM I. *Let X be a Hilbert space and $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\bar{x}$ be a local minimizer of f and consider the following statements:*

(i) $0 \in \partial f(\bar{x})$ *and the generalized second order subdifferential $\partial^2 f(\bar{x}, 0)$ is positively definite.*
(ii) $\bar{x}$ *is a stable 2-order sharp minimizer of f.*
(iii) $\bar{x}$ *is a tilt-stable 1-order minimizer of f.*

Then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) *when f is subdifferentially continuous and proximally regular, while* (ii) *and* (iii) *are always equivalent when X is finite dimensional.*

Note that Theorem I(i) has no counterpart in the general case when q is any number in $(1, +\infty) \setminus \{2\}$, mainly due to the fact that we do not have a satisfactory notion/theory for the corresponding higher order subdifferentials (especially no "fractional-order" subdifferentials have been considered). This may be a reason why no author has considered the general case of $q \in (1, +\infty)$ up to now. In the line of the equivalence of (ii) and (iii) in Theorem I, one of the main goals of the present paper is to establish the following result: $\bar{x}$ is a tilt-stable p-order (with $p \in (0, +\infty)$) minimizer of a proper lower semicontinuous function f on a Banach space if and only if $\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f.

Metric regularity and strong metric regularity for a multifunction F between two Banach spaces are becoming an important and active area of research in variational analysis and optimization theory (cf. [4, 7, 15, 19, 26, 29, 33, 34]). It is of particular interest that the multifunction F is taken as the subdifferential mapping of a proper lower semicontinuous function f. Under the assumption that f is a proper lower semicontinuous convex function on a Hilbert space X, Artacho and Geoffroy [1] first proved that $\bar{x} \in \mathrm{dom}(f)$ is a stable 2-order strong minimizer of f if and only if its subdifferential mapping ∂f is strongly metrically regular at $\bar{x}$ for 0. Afterward, relaxing the convexity of f to the assumption that f is subdifferentially continuous and proximally regular, Drusvyatskiy and Lewis [8] proved that the corresponding equivalence still holds in the finite dimensional case. Very recently, these works have been pushed further by Mordukhovich and Nghia [20, 22] and Mordukhovich and Rockafellar [24] for the case when X is an Asplund or Hilbert space. The present paper mainly concerns the more general Hölder situation. For general $p \in (0, +\infty)$, in terms of p-order metric/p-order strong metric regularity (for their definitions see section 2) of the subdifferential mapping ∂f, we further study the Hölder strong stable minimizers of f.

The rest of the paper is organized as follows. Section 2 provides some notions in variational analysis and preliminary results (some of them are new and are of interest by themselves). In section 3, we provide dual techniques for $C^{1,p}$ smooth conjugate functions, which together with techniques of variational analysis play an important role in the proofs for some of our main results. For a local minimizer $\bar{x}$ of a proper lower semicontinuous function f on a Banach space, any $p \in (0, +\infty)$, and $q = \frac{1+p}{p}$, we prove in section 4 that $\bar{x}$ is a q-order sharp minimizer of f if ∂f is p-order strongly metrically subregular at $\bar{x}$ for 0 and that $\bar{x}$ is not necessarily a q-order weak sharp minimizer of f if ∂f is p-order metrically subregular at $\bar{x}$ for 0; this is a reason why the paper mainly considers the Hölder sharp minimizer (not the Hölder weak sharp minimizer). However, the main aim of this section is to consider the relationship between the stable q-order sharp minimizer of f and the order p strong metric regularity of ∂f. Section 5 is devoted to the Hölder stable minimizers, Hölder tilt-stable minimizers, and Hölder metric regularity of the corresponding subdifferential mapping. We prove that $\bar{x}$ is a stable q-order sharp minimizer of f if and only if $\bar{x}$ is a tilt-stable p-order minimizer of f. These properties are further studied in terms of the metric regularity of the subdifferential mapping ∂f and the $C^{1,p}$-smoothness of the conjugate function $(f+\delta_{B_X[\bar{x},r]})^*$.

2. Preliminaries. Let X be a Banach space with the topological dual X^*. For a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, the Clarke–Rockafellar subdifferential $\partial f(\bar{x})$ of f at $\bar{x} \in \mathrm{dom}(f)$ is defined as

$$\partial f(\bar{x}) := \{x^* \in X^* | \langle x^*, h\rangle \leq f^{\uparrow}(\bar{x}, h) \quad \forall h \in X\},$$

where

$$f^{\uparrow}(\bar{x}, h) := \lim_{\varepsilon\downarrow 0} \limsup_{x \xrightarrow{f} \bar{x}, t\downarrow 0} \inf_{w\in h+\varepsilon B_X} \frac{f(x+tw)-f(x)}{t}.$$

In the case when f is locally Lipschitzian around $\bar{x}$, $f^{\uparrow}(\bar{x}, h)$ reduces to the Clarke directional derivative

$$f^{\circ}(\bar{x}, h) := \limsup_{t\to 0^+, x\to\bar{x}} \frac{f(x+th)-f(x)}{t}.$$

It is well known that if f is convex, then

$$\partial f(\bar{x}) = \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \leq f(x) - f(\bar{x}) \quad \forall x \in X\}.$$

In the following definition we recall two important and useful notions in variational analysis (cf. [8, 9, 20]).

Definition 2.1. *Let f be a proper lower semicontinuous function on a Banach space X and $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(\partial f)$, where $\mathrm{gph}(\partial f)$ denotes the graph of ∂f. We say that*

(i) *f is prox-regular at $\bar{x}$ for $\bar{x}^*$ if there exist $\sigma, \delta \in (0, +\infty)$ such that for all $u \in B(\bar{x}, \delta)$ with $|f(u) - f(\bar{x})| < \delta$ and $u^* \in \partial f(u) \cap B_{X^*}(\bar{x}^*, \delta)$ one has*

$$\langle u^*, x-u\rangle \leq f(x) - f(u) + \sigma\|x-u\|^2 \quad \forall x \in B_X(\bar{x}, \delta);$$

(ii) *f is said to be subdifferentially continuous at $\bar{x}$ for $\bar{x}^*$ if $\{f(x_n)\}$ converges to $f(\bar{x})$ whenever a sequence $\{(x_n, x_n^*)\}$ in $\mathrm{gph}(\partial f)$ converges to $(\bar{x}, \bar{x}^*)$.*

It is known (cf. [33]) that if g is a $C^{1,1}$ mapping between Banach spaces X and Y and $\phi : Y \to \overline{\mathbb{R}}$ is a proper lower semicontinuous convex function satisfying the Robinson qualification condition at $\bar{x}$ in the sense that

$$\mathbb{R}_+(\mathrm{dom}(\phi) - g(\bar{x})) - \nabla g(\bar{x})(X) = Y,$$

then the convex-composite function $\phi \circ g$ is prox-regular and subdifferentially continuous at $\bar{x}$ for any $x^* \in \partial(\phi \circ g)(\bar{x})$.

Recall that a set A in $X \times X^*$ is said to be

(a) monotone if

$$0 \leq \langle x_1^* - x_2^*, x_1 - x_2\rangle \quad \forall (x_1, x_1^*), (x_2, x_2^*) \in A;$$

(b) maximal monotone if A is monotone and $A = B$ whenever a monotone subset B of $X \times X^*$ satisfies $A \subset B$.

A multifunction $M : X \rightrightarrows X^*$ is said to be monotone (resp., maximal monotone) if $\mathrm{gph}(M)$ is a monotone (resp., maximal monotone) set in $X \times X^*$. We also need the following notion related to monotonicity.

Definition 2.2. *We say that a multifunction $F : X \rightrightarrows X^*$ is locally maximal monotone at $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(F)$ if each neighborhood V of $(\bar{x}, \bar{x}^*)$ contains a neighborhood W of $(\bar{x}, \bar{x}^*)$ such that $\mathrm{gph}(F) \cap W$ is a maximal monotone subset of W, that is, $\mathrm{gph}(F) \cap W$ is monotone and $\mathrm{gph}(F) \cap W = \mathrm{gph}(B)$ whenever $B : X \rightrightarrows X^*$ is monotone such that $\mathrm{gph}(F) \cap W \subset \mathrm{gph}(B) \subset W$.*

Some existing papers on the tilt stability adopt the local maximal monotonicity of F at $(\bar{x}, \bar{x}^*)$ in the following weaker sense: *there exists a neighborhood V_0 of $(\bar{x}, \bar{x}^*)$ such that $\mathrm{gph}(F) \cap V_0$ is a maximal monotone subset of V_0.* However, should this weaker notion be adopted, the proofs given in [28, 20] appear to have a gap. This is why we introduce the stronger notion in Definition 2.2. The following is a lemma on the local maximal monotonicity which would be sufficient for our purpose here. When X is a Hilbert space we identify X^* with X as usual.

Lemma 2.3. *Let X be a Hilbert space and $A : X \rightrightarrows X$ be a monotone operator. Suppose that $\bar{x} \in X$, $\bar{y} \in A(\bar{x})$ and $\sigma \in (0, +\infty)$ are such that $\bar{y} + \sigma\bar{x} \in \mathrm{int}((A + \sigma I)(X))$. Then, $A + \sigma I$ is locally maximal monotone at $(\bar{x}, \bar{y} + \sigma\bar{x})$. Consequently, if A is maximal monotone, then $A + \sigma I$ is locally maximally monotone at any point in $\mathrm{gph}(A + \sigma I)$.*

Proof. Recall (cf. [5, Theorem 21.1]) that whenever A is maximal monotone, $(A + \sigma I)(X) = X$ for all $\sigma > 0$. We need only prove the first assertion of this lemma. Let $v_i \in X$ and $x_i \in (A + \sigma I)^{-1}(v_i)$ $(i = 1, 2)$. Then $v_i \in (A + \sigma I)(x_i) = A(x_i) + \sigma x_i$, that is, $v_i - \sigma x_i \in A(x_i)$. This and the monotonicity of A imply that

$$0 \le \langle v_2 - \sigma x_2 - (v_1 - \sigma x_1), x_2 - x_1\rangle = \langle v_2 - v_1, x_2 - x_1\rangle - \sigma\|x_2 - x_1\|^2.$$

It follows that $\|x_2 - x_1\| \le \frac{1}{\sigma}\|v_2 - v_1\|$. This means that $(A + \sigma I)^{-1}$ is single-valued on X and

$$\|(A + \sigma I)^{-1}(v_1) - (A + \sigma I)^{-1}(v_2)\| \le \sigma^{-1}\|v_1 - v_2\| \ \ \forall v_1, v_2 \in (A + \sigma I)(X). \tag{2.1}$$

Let V be a neighborhood of $(\bar{x}, \bar{y} + \sigma\bar{x})$. Then, there exist $r_1, r_2 \in (0, +\infty)$ such that

$$B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2) \subset V \ \text{ and } \ B(\bar{y} + \sigma\bar{x}, r_2) \subset (A + \sigma I)(X) \tag{2.2}$$

(thanks to the assumption that $\bar{y} + \sigma\bar{x} \in \mathrm{int}((A + \sigma I)(X))$). By (2.1), we assume without loss of generality that

$$(A + \sigma I)^{-1}(B(\bar{y} + \sigma\bar{x}, r_2)) \subset B(\bar{x}, r_1). \tag{2.3}$$

We only need to show that $\mathrm{gph}(A + \sigma I) \cap (B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2))$ is a maximal monotone subset of $B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2)$. To do this, suppose to the contrary that there exists $(x_0, y_0) \in B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2)$ such that

$$0 \le \langle v - y_0, u - x_0\rangle \ \ \forall (u, v) \in \mathrm{gph}(A + \sigma I) \cap (B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2)) \tag{2.4}$$

and $(x_0, y_0) \notin \mathrm{gph}(A + \sigma I)$. Thus, $h_0 := (A + \sigma I)^{-1}(y_0) - x_0 \ne 0$. Take a sequence $\{t_n\} \subset (0, +\infty)$ convergent to 0 such that each $y_0 - \frac{t_n h_0}{\|h_0\|}$ lies in the open ball $B(\bar{y} + \sigma\bar{x}, r_2)$. By (2.2), let $u_n := (A + \sigma I)^{-1}(y_0 - \frac{t_n h_0}{\|h_0\|})$. Then, by (2.3), one has

$$\left(u_n, y_0 - \frac{t_n h_0}{\|h_0\|}\right) \in \mathrm{gph}(A + \sigma I) \cap (B(\bar{x}, r_1) \times B(\bar{y} + \sigma\bar{x}, r_2)).$$

It follows from (2.4) that

$$0 \le \left\langle -\frac{t_n h_0}{\|h_0\|}, u_n - x_0 \right\rangle = -\frac{t_n}{\|h_0\|}\left(\langle h_0, h_0\rangle + \langle h_0, u_n - x_0 - h_0\rangle\right)$$
$$= -\frac{t_n}{\|h_0\|}\left(\|h_0\|^2 + \left\langle h_0, (A+\sigma I)^{-1}\left(y_0 - \frac{t_n h_0}{\|h_0\|}\right) - (A+\sigma I)^{-1}(y_0)\right\rangle\right).$$

This together with (2.1) implies that

$$\|h_0\|^2 \le -\left\langle h_0, (A+\sigma I)^{-1}\left(y_0 - \frac{t_n h_0}{\|h_0\|}\right) - (A+\sigma I)^{-1}(y_0)\right\rangle \le \sigma^{-1}\|t_n h_0\|,$$

contradicting $t_n \to 0$ and $h_0 \neq 0$. The proof is completed. □

The following known lemma (cf. [3, Theorem 3.3]) is useful for our later analysis.

LEMMA 2.4. *Let f be a proper lower semicontinuous function on a Hilbert space X and let $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(\partial f)$. Suppose that f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^*$. Then ∂f is hypomonotone around $(\bar{x}, \bar{x}^*)$ in the sense that there exist $\delta, \sigma \in (0, +\infty)$ such that*

$$0 \le \langle x_1^* - x_2^*, x_1 - x_2\rangle + \sigma\|x_1 - x_2\|^2 \quad \forall (x_1, x_1^*), (x_2, x_2^*) \in \mathrm{gph}(\partial f) \cap (B(\bar{x}, \delta) \times B(\bar{x}^*, \delta)).$$

For $q \in (1, +\infty)$, we define the q-order generalized directional derivative of f at $\bar{x} \in X$ for $x^* \in \partial f(\bar{x})$ as follows:

$$d^q f(\bar{x}, x^*)(h) := \liminf_{t\to 0^+, h'\to h} \frac{f(\bar{x} + th') - f(\bar{x}) - \langle x^*, th'\rangle}{t^q} \quad \forall h \in X.$$

Clearly, $d^q f(\bar{x}, x^*)$ is positively q-homogeneous, that is,

$$d^q f(\bar{x}, x^*)(\alpha h) = \alpha^q d^q f(\bar{x}, x^*)(h) \quad \forall (\alpha, h) \in (0, +\infty) \times X.$$

It is easy to verify that if f is twice smooth around $\bar{x}$, then

$$d^2 f(\bar{x}, \nabla f(\bar{x}))(h) = \frac{1}{2}\nabla^2 f(\bar{x})(h^2) \quad \forall h \in X.$$

For $\bar{x} \in \mathrm{dom}(f)$ and $x^* \in \partial f(\bar{x})$, we say that the q-order generalized directional derivative $d^q f(\bar{x}, x^*)$ is strictly positive if

$$d^q f(\bar{x}, x^*)(h) > 0 \quad \forall h \in X \setminus \{0\}.$$

In the finite dimensional case, we has the following result on the q-order generalized directional derivative.

PROPOSITION 2.5. *Let X be a finite dimensional Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $q \in (1, +\infty)$, $\bar{x} \in \mathrm{dom}(f)$, and $x^* \in \partial f(\bar{x})$. Then $d^q f(\bar{x}, x^*)$ is strictly positive if and only if there exists $\eta > 0$ such that*

$$d^q f(\bar{x}, x^*)(h) \ge \eta\|h\|^q \quad \forall h \in X. \tag{2.5}$$

Proof. Clearly, (2.5) implies the strict positivity of $d^q f(\bar{x}, x^*)$. To prove the converse implication, suppose to the contrary that there exists a sequence $\{h_n\}$ in X such that

$$\|h_n\| = 1 \text{ and } d^q f(\bar{x}, x^*)(h_n) < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

(thanks to the positive q-homogeneity of $d^q f(\bar{x}, x^*)$). Hence, there exist sequences $\{u_n\}$ in $X \setminus \{0\}$ with $\|u_n - h_n\| \to 0$ and $\{t_n\}$ in $(0, +\infty)$ convergent to 0 such that

$$\frac{f(\bar{x} + t_n u_n) - f(\bar{x}) - \langle x^*, t_n u_n \rangle}{t_n^q} < \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Since X is finite dimensional, we assume without loss of generality that $u_n \to u$ with $\|u\| = 1$. Thus, by (2.6), one has $d^q f(\bar{x}, x^*)(u) \leq 0$, contradicting the strict positivity of $d^q f(\bar{x}, x^*)$. The proof is complete. ∎

To conclude this section, we introduce the notions of Hölder metric regularity/subregularity for multifunctions. Let F be a multifunction between two Banach spaces X and Y. For $p \in (0, +\infty)$ and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, we say that F is

(i) p-order metrically regular at $\bar{x}$ for $\bar{y}$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))^p \quad \forall (x, y) \in B_X(\bar{x}, \delta) \times B_Y(\bar{y}, \delta); \tag{2.7}$$

(ii) p-order strongly metrically regular at $\bar{x} \in X$ for $\bar{y} \in F(\bar{x})$ if there exist $\tau, \delta, \eta \in (0, +\infty)$ such that *(2.7)* holds and $F^{-1}(y) \cap B_X(\bar{x}, \eta)$ is a singleton for each $y \in B_Y(\bar{y}, \delta)$;

(iii) p-order (strongly) metrically subregular at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ if there exist $\tau, \delta \in (0, +\infty)$ such that $(F^{-1}(\bar{y}) \cap B_X(\bar{x}, \delta) = \{\bar{x}\}$ and)

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x))^p \quad \forall x \in B_X(\bar{x}, \delta).$$

On one hand, some results in connection with geometric control theory and mathematical programming are based on the Hölder metric regularity/subregularity assumptions, but on the other hand, there are only a few studies of p-order metric regularity/subregularity for $p \in (0, +\infty) \setminus \{1\}$ (cf. [11, 13, 16, 18]). For $p = 1$, metric regularity/subregularity have been well studied (cf. [4, 7, 15, 19, 26, 29, 33, 34]). The following lemma is convenient for our later analysis.

LEMMA 2.6. *Let F be a multifunction between two Banach spaces X and Y. Let $p \in (0, +\infty)$ and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. Then F is p-order strongly metrically regular at $\bar{x} \in X$ for $\bar{y}$ if and only if there exist $\kappa, r \in (0, +\infty)$ such that for any $v \in B_Y(\bar{y}, r)$ there exists $x_v \in F^{-1}(v)$, with $x_{\bar{y}} = \bar{x}$, satisfying*

$$\kappa \|x - x_v\| \leq d(v, F(x))^p \quad \forall x \in B_X(\bar{x}, r). \tag{2.8}$$

If, in addition, $p > 1$, then F is p-order strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if there exist $\eta, r, \kappa \in (0, +\infty)$ such that

$$F^{-1}(v) \cap B(\bar{x}, \eta) = \{\bar{x}\} \text{ and } \kappa \|x - \bar{x}\| \leq d(v, F(x))^p \tag{2.9}$$

for all $(x, v) \in B_X(\bar{x}, r) \times B_Y(\bar{y}, r)$.

Proof. First suppose that F is p-order strongly metrically regular at $\bar{x} \in X$ for $\bar{y}$. Take $\tau, \delta, \eta \in (0, +\infty)$ such that (2.7) holds and $F^{-1}(y) \cap B_X(\bar{x}, \eta)$ is a singleton for each $y \in B_Y(\bar{y}, \delta)$. By (2.7), one has

$$d(\bar{x}, F^{-1}(y)) \leq \tau d(y, F(\bar{x}))^p \leq \tau \|y - \bar{y}\|^p \quad \forall y \in B_Y(\bar{y}, \delta).$$

Hence, for each $v \in B_Y(\bar{y}, \delta)$ there exists $x_v \in F^{-1}(v)$ with $x_{\bar{y}} = \bar{x}$ such that $\|\bar{x} - x_v\| \leq 2\tau \|v - \bar{y}\|^p$, and so

$$F^{-1}(v) \cap B_X(\bar{x}, \eta) = \{x_v\} \quad \forall v \in B_Y\left(\bar{y}, \min\left\{\delta, \left(\frac{\eta}{2\tau}\right)^{\frac{1}{p}}\right\}\right). \tag{2.10}$$

Let $v \in B_Y(\bar{y}, \min\{\delta, (\frac{\delta}{8\tau})^{\frac{1}{p}}, (\frac{\eta}{8\tau})^{\frac{1}{p}}\})$, and set $\eta_0 := \min\{\eta, \delta\}$. Then,

$$\|\bar{x} - x_v\| < \frac{\eta_0}{4} \text{ and } B_X\left(\bar{x}, \frac{\eta_0}{4}\right) \subset B_X\left(x_v, \frac{3\eta_0}{4}\right) \subset B_X(\bar{x}, \delta).$$

This and (2.10) imply that

$$d(x, F^{-1}(v)) = d\left(x, F^{-1}(v) \cap B_X\left(x, \frac{\eta_0}{2}\right)\right) = \|x - x_v\| \quad \forall x \in B_X\left(\bar{x}, \frac{\eta_0}{4}\right).$$

It follows from (2.7) that

$$\|x - x_v\| \le \tau d(x, F(v))^p \quad \forall x \in B_X\left(\bar{x}, \frac{\eta_0}{4}\right).$$

This shows that (2.8) holds with $\kappa = \frac{1}{\tau}$ and $r = \min\{\frac{\eta_0}{4}, (\frac{\delta}{8\tau})^{\frac{1}{p}}, (\frac{\eta}{8\tau})^{\frac{1}{p}}\}$, and so the necessity part holds.

Suppose that there exist $\kappa, r \in (0, +\infty)$ such that for each $v \in B_Y(\bar{y}, r)$ there exists $x_v \in F^{-1}(v)$, with $x_{\bar{y}} = \bar{x}$, satisfying (2.8). Thus, (2.7) holds with $\tau = \frac{1}{\kappa}$ and $\delta = r$, and

$$\kappa\|\bar{x} - x_v\| \le d(v, F(\bar{x}))^p \le \|v - \bar{y}\|^p \quad \forall v \in B_Y(\bar{y}, r); \tag{2.11}$$

it follows that $x_v \in B_X(\bar{x}, r)$ for all $v \in B_Y(\bar{y}, \min\{r, (\kappa r)^{\frac{1}{p}}\})$. This and (2.8) imply that

$$F^{-1}(v) \cap B_X(\bar{x}, r) = \{x_v\} \quad \forall v \in B_Y\left(\bar{y}, \min\left\{r, (\kappa r)^{\frac{1}{p}}\right\}\right). \tag{2.12}$$

Therefore, the sufficiency part holds.

In the case when $p > 1$, it suffices to show that (2.8), (2.11), and (2.12) imply that there exists $r' > 0$ such that $x_v = \bar{x}$ for all $v \in B_Y(\bar{y}, r')$. From (2.8) and (2.11), it is easy to verify that there exists sufficiently small $r' > 0$ such that

$$\kappa\|x_u - x_v\| \le \|u - v\|^p \quad \forall u, v \in B_Y(\bar{y}, r').$$

Since $p > 1$, it follows that the function $v \mapsto x_v$ is Fréchet differentiable on $B_Y(\bar{y}, r_1)$ and its derivative is constantly 0 on $B_Y(\bar{y}, r')$. Therefore, $x_v = x_{\bar{y}} = \bar{x}$ for all $v \in B_Y(\bar{x}, r')$. The proof is complete. ∎

Regarding subregularity instead of regularity in Lemma 2.6, one has the following corresponding result (we omit its easy proof): F is p-order strongly metrically subregular at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ if and only if there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa\|x - \bar{x}\| \le d(\bar{y}, F(x))^p \quad \forall x \in B_X(\bar{x}, \delta).$$

3. Smooth conjugate functions. The most useful notion in duality theory is undoubtedly the conjugate function of a proper lower semicontinuous function, which has been well studied. In this section, we provide two interesting properties of differentiable conjugate functions which will play an important role in the proofs of our main results. Let f be a proper lower semicontinuous function on a Banach space X and recall that the conjugate function f^* of f is a weak* lower semicontinuous convex function on X^* such that

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\} = -\inf\{f_{x^*}(x) : x \in X\} \ \forall x^* \in X^*,$$

where f_{x^*} is as in (1.5). In the case when f is bounded below by a continuous linear functional (i.e., $\mathrm{dom}(f^*) \neq \emptyset$), one can define the closed convex envelope $f_{\overline{\mathrm{co}}} : X \to \overline{\mathbb{R}}$ of f as

$$\mathrm{epi}(f_{\overline{\mathrm{co}}}) = \overline{\mathrm{co}(\mathrm{epi}(f))};$$

it is known and easy to verify that $f_{\overline{\mathrm{co}}}$ is a proper lower semicontinuous convex function such that

$$f^*_{\overline{\mathrm{co}}} = f^*, \ f_{\overline{\mathrm{co}}}(x) \leq f(x), \ \text{ and } \ \langle x^*, x\rangle - f^*(x^*) \leq f_{\overline{\mathrm{co}}}(x) \ \ \forall (x, x^*) \in X \times X^*.$$

For $x^* \in X^*$ and $x \in X$, it is known and easy to verify that

$$f^*(x^*) = \langle x^*, x\rangle - f(x) \Longrightarrow x \in \partial f^*(x^*) \text{ and } f_{\overline{\mathrm{co}}}(x) = f(x)$$

and

$$x^* \in \partial f_{\overline{\mathrm{co}}}(x) \Longleftrightarrow x \in \partial f^*(x^*).$$

The following inequality is well known and often used in duality theory:

$$f^*(u^*) \geq \langle u^*, u\rangle - f(u) \quad \forall (u^*, u) \in X^* \times X.$$

A better (and useful for our later analysis) inequality is shown in the following proposition under the $C^{1,p}$ smoothness assumption. For a proper lower semicontinuous function ϕ on a Banach space Z and $p > 0$, recall that ϕ is $C^{1,p}$ smooth on a subset V of Z if ϕ is differentiable at every point of V and

$$\|\nabla\phi(z_1) - \nabla\phi(z_2)\| \leq L\|z_1 - z_2\|^p \ \text{ for all } z_1, z_2 \in V \text{ and some } L \in [0, +\infty).$$

PROPOSITION 3.1. *Let D be an open subset of a Banach space E, and let $g : E \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Suppose that g is differentiable on D and that there exist $p, \kappa \in (0, +\infty)$ such that*

$$\|\nabla g(x_1) - \nabla g(x_2)\| \leq \kappa\|x_1 - x_2\|^p \quad \forall x_1, x_2 \in D. \tag{3.1}$$

Let $\bar{u} \in D$ and $\delta > 0$ be such that

$$B_E(\bar{u}, (1 + 2^{\frac{1}{p}})\delta) \subset D. \tag{3.2}$$

Then

$$g^*(x^*) \geq \langle x^*, u\rangle - g(u) + \frac{p}{(1+p)\kappa^{\frac{1}{p}}}\|x^* - \nabla g(u)\|^{\frac{1+p}{p}} \tag{3.3}$$

for all $(u, x^*) \in B_E(\bar{u}, \delta) \times B_{E^*}(\nabla g(\bar{u}), \kappa\delta^p)$.

Proof. Let $\delta_0 := (1 + 2^{\frac{1}{p}})\delta$. Then, (3.1) and (3.2) imply that, for any $u \in B(\bar{u}, \delta)$ and any $v \in B(\bar{u}, \delta_0)$,

$$\begin{aligned} g(v) - g(u) - \langle \nabla g(u), v - u\rangle &= \int_0^1 \langle \nabla g(u + t(v-u)) - \nabla g(u), v - u\rangle dt \\ &\leq \int_0^1 \kappa t^p \|v - u\|^{1+p} dt \\ &= \frac{\kappa\|v - u\|^{1+p}}{1+p}. \end{aligned}$$

Let $(u, x^*) \in B(\bar{u}, \delta) \times B(\nabla g(\bar{u}), \kappa\delta^p)$. Then

$$\begin{aligned} g^*(x^*) &\geq \sup_{v \in B(\bar{u},\delta_0)} (\langle x^*, v\rangle - g(v)) \\ &\geq \sup_{v \in B(\bar{u},\delta_0)} \left(\langle x^*, v\rangle - g(u) - \langle \nabla g(u), v-u\rangle - \frac{\kappa\|v-u\|^{1+p}}{1+p} \right) \\ &= \langle x^*, u\rangle - g(u) + \sup_{v \in B(\bar{u},\delta_0)} \left(\langle x^* - \nabla g(u), v-u\rangle - \frac{\kappa\|v-u\|^{1+p}}{1+p} \right). \end{aligned}$$

Let

$$\zeta := \sup_{v \in B(\bar{u},\delta_0)} \left(\langle x^* - \nabla g(u), v-u\rangle - \frac{\kappa\|v-u\|^{1+p}}{1+p} \right). \tag{3.4}$$

To prove (3.3), it suffices to show that

$$\zeta \geq \frac{p}{(1+p)\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1+p}{p}}. \tag{3.5}$$

Take a sequence $\{z_n\}$ with each $\|z_n\| = 1$ such that

$$\langle x^* - \nabla g(u), z_n\rangle \to \|x^* - \nabla g(u)\|. \tag{3.6}$$

For each $n \in \mathbb{N}$, let

$$v_n := u + \frac{1}{\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1}{p}} z_n.$$

Then

$$\begin{aligned} \|v_n - \bar{u}\| &\leq \|u - \bar{u}\| + \frac{1}{\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u))\|^{\frac{1}{p}} \\ &< \delta + \frac{1}{\kappa^{\frac{1}{p}}} (\|x^* - \nabla g(\bar{u})\| + \|\nabla g(\bar{u}) - \nabla g(u)\|)^{\frac{1}{p}} \\ &\leq \delta + \frac{1}{\kappa^{\frac{1}{p}}} (\kappa\delta^p + \kappa\|\bar{u} - u\|^p)^{\frac{1}{p}} \\ &\leq \delta(1 + 2^{\frac{1}{p}}) = \delta_0. \end{aligned}$$

It follows from (3.4) that

$$\begin{aligned} \zeta &\geq \langle x^* - \nabla g(u), v_n - u\rangle - \frac{\kappa\|v_n - u\|^{1+p}}{1+p} \\ &= \frac{1}{\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1}{p}} \langle x^* - \nabla g(u), z_n\rangle - \frac{1}{(1+p)\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1+p}{p}}. \end{aligned}$$

This and (3.6) imply that

$$\begin{aligned} \zeta &\geq \frac{1}{\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1+p}{p}} - \frac{1}{(1+p)\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1+p}{p}} \\ &= \frac{p}{(1+p)\kappa^{\frac{1}{p}}} \|x^* - \nabla g(u)\|^{\frac{1+p}{p}}, \end{aligned}$$

verifying (3.5). The proof is complete. ∎

Remark. In the case when $p > 1$, (3.1) implies that $\triangledown g$ is constant on any connected component of D, and so $g(x) = \langle \triangledown g(\bar{u}), x - \bar{u}\rangle + g(\bar{u})$ for all $x \in B_E(\bar{u}, \delta)$ whenever $B_E(\bar{u}, \delta) \subset D$.

If an extended real-valued function g on the dual space X^* is Fréchet differentiable at $x^* \in \mathrm{dom}(g)$, then its derivative $\triangledown g(x^*)$ is a norm-continuous linear functional on X^* (i.e., $\triangledown g(x^*) \in X^{**}$). The continuity conclusion is strengthened in the next proposition for the case when g is assumed to be the conjugate function f^* of some proper lower semicontinuous function f on X, which will be useful for our analysis later.

PROPOSITION 3.2. *Let f be a proper lower semicontinuous function on a Banach space X such that its conjugate function f^* is Fréchet differentiable at $x^* \in X^*$. Then the derivative $\triangledown f^*(x^*)$ is a weak* continuous linear functional on X^*, that is, $\triangledown f^*(x^*) \in X$.*

Proposition 3.2 was established by Asplund and Rockafellar [2] (also see [31, Corollary 3.3.4]).

4. Hölder metric regularity and Hölder minima. In this section, we consider the Hölder sharp minimizers and the stable Hölder sharp minimizers. The consideration will be mainly carried out in terms of the Hölder strong metric subregularity/regularity of the subdifferential mapping ∂f.

The following theorem describes the Hölder sharp minimizer in terms of the Hölder strong metric subregularity of the subdifferential mapping. The quantitative formula (4.4) will play an important role for the corresponding stability result given in Theorem 4.3.

THEOREM 4.1. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ and $p \in (0, +\infty)$. The following statements hold:*

(i) *Suppose that there exist $r, \kappa, \delta \in (0, \infty)$ such that*

$$\min_{x \in B_X[\bar{x}, r]} f(x) = f(\bar{x}) \tag{4.1}$$

and

$$\kappa \|x - \bar{x}\| \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta). \tag{4.2}$$

Then

$$\tau \|x - \bar{x}\|^{\frac{1+p}{p}} \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \eta), \tag{4.3}$$

where

$$\tau := \frac{p\kappa^{\frac{1}{p}}}{(1+p)^{\frac{1+p}{p}}} \quad \text{and} \quad \eta := \frac{1+p}{1+2p} \min\{r, \delta\}. \tag{4.4}$$

(ii) *Suppose that f is convex and there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\tau \|x - \bar{x}\|^{\frac{1+p}{p}} \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta).$$

Then

$$\tau^p \|x - \bar{x}\| \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta).$$

Consequently, if f is convex, then ∂f is p-order strongly metrically subregular at $\bar{x}$ for 0 if and only if $\bar{x}$ is a $\frac{1+p}{p}$-order sharp minimizer of f.

Proof. Statement (ii) is immediate from [31, Corollary 3.3.4 (i)⇒(iv)] with $\psi(t) = \tau t^{\frac{1+p}{p}}$. We only need to prove (i). To do this, suppose to the contrary that (4.3) is not true, namely, there exists $x_0 \in B(\bar{x}, \eta)$ such that $\tau\|x_0 - \bar{x}\|^{\frac{1+p}{p}} > f(x_0) - f(\bar{x})$. This means, by (4.1), that

$$f(x_0) < \min_{x\in B_X[\bar{x},r]} f(x) + \tau\|x_0 - \bar{x}\|^{\frac{1+p}{p}}.$$

Take a $\tau' \in (0,\ \tau)$ sufficiently close to τ such that

$$f(x_0) < \min_{x\in B_X[\bar{x},r]} f(x) + \tau'\|x_0 - \bar{x}\|^{\frac{1+p}{p}}.$$

Then, by the Ekeland variational principle, there exists $u \in B_X[\bar{x}, r]$ such that

$$\|u - x_0\| < \frac{p}{1+p}\|x_0 - \bar{x}\| \tag{4.5}$$

and

$$f(u) \le f(x) + \frac{(1+p)\tau'\|x_0 - \bar{x}\|^{\frac{1}{p}}}{p}\|x - u\| \quad \forall x \in B_X[\bar{x}, r]. \tag{4.6}$$

Thus, by (4.5) and the choice of x_0, we have

$$\|u - \bar{x}\| \le \|u - x_0\| + \|x_0 - \bar{x}\| \le \frac{1+2p}{1+p}\|x_0 - \bar{x}\| < \frac{(1+2p)\eta}{1+p}.$$

It follows from (4.4) that

$$u \in B_X(\bar{x}, r) \cap B_X(\bar{x}, \delta) \tag{4.7}$$

and so one can make use of (4.6) to get

$$0 \in \partial\left(f + \frac{(1+p)\tau'\|x_0 - \bar{x}\|^{\frac{1}{p}}}{p}\|\cdot - u\|\right)(u) \subset \partial f(u) + \frac{(1+p)\tau'\|x_0 - \bar{x}\|^{\frac{1}{p}}}{p}B_{X^*},$$

where B_{X^*} is the closed unit ball of X^*. Hence, there exists $u^* \in B_{X^*}$ such that $y^* := \frac{(1+p)\tau'\|x_0 - \bar{x}\|^{\frac{1}{p}}u^*}{p} \in \partial f(u)$. This, together with (4.2) and (4.7), implies that

$$\kappa\|u - \bar{x}\| \le \|y^*\|^p \le \left(\frac{(1+p)\tau'}{p}\right)^p \|x_0 - \bar{x}\|.$$

Noting (by (4.5)) that

$$\|u - \bar{x}\| \ge \|x_0 - \bar{x}\| - \|u - x_0\| \ge \frac{\|x_0 - \bar{x}\|}{1+p},$$

it follows that $\frac{\kappa\|x_0 - \bar{x}\|}{1+p} \le (\frac{(1+p)\tau'}{p})^p\|x_0 - \bar{x}\|$. Therefore, $\tau' \ge \frac{p\kappa^{\frac{1}{p}}}{(1+p)^{\frac{1+p}{p}}} = \tau$, contradicting the choice of τ'. This shows that (4.3) holds. The proof is complete. ∎

The following example shows that the corresponding assertion for weak sharp minimizers (rather than for sharp minimizers) in Theorem 4.1(i) is not valid: a local minimizer $\bar{x}$ of f is not necessarily a $\frac{1+p}{p}$-order weak sharp minimizer of f even though ∂f is p-order metrically subregular at $\bar{x}$ for 0; this is a reason why we mainly consider Hölder sharp minimizers and stable Hölder sharp minimizers.

Let $f:\mathbb{R}\to\mathbb{R}$ be such that

$$f(x)=\begin{cases} 0 & \text{if } x\le 0,\\ x-\frac{1}{2^n}+\frac{1}{2^{n^2}} & \text{if } \frac{1}{2^n}\le x<\frac{1}{2^n}+\frac{1}{2^{(n-1)^2}}-\frac{1}{2^{n^2}} \quad (\forall n\in\mathbb{N}_1),\\ \frac{1}{2^{(n-1)^2}} & \text{if } \frac{1}{2^n}+\frac{1}{2^{(n-1)^2}}-\frac{1}{2^{n^2}}\le x<\frac{1}{2^{n-1}} \quad (\forall n\in\mathbb{N}_1),\\ \frac{1}{2} & \text{if } \frac{1}{2}\le x, \end{cases}$$

where $\mathbb{N}_1:=\mathbb{N}\setminus\{1,2\}$. Then f is a nonnegative Lipschitz function and

$$S:=\left\{x\in\mathbb{R}:\ f(x)=\inf_{u\in\mathbb{R}}f(x)\right\}=-\mathbb{R}_+.$$

Hence, for any $q\in(0,\ +\infty)$, $\lim\limits_{n\to\infty}\frac{f(\frac{1}{2^n})}{d(\frac{1}{2^n},S)^q}=\lim\limits_{n\to\infty}\frac{1}{2^{n^2-qn}}=0$; this implies that 0 is not a q-order weak sharp minimizer of f. Next, we show that ∂f is p-order metrically subregular at $(0,0)$ for any $p\in(0,\ +\infty)$. Indeed, it is easy from the definition of f to verify that

$$\partial f(x)=\begin{cases} [0,\ 1] & \text{if } x\in\{0\}\cup\{\frac{1}{2^n}:\ n\in\mathbb{N}_1\}\cup\{\frac{1}{2^n}+\frac{1}{2^{(n-1)^2}}-\frac{1}{2^{n^2}}:\ n\in\mathbb{N}_1\},\\ \{1\} & \text{if } x\in\bigcup\limits_{n\in\mathbb{N}_1}\left(\frac{1}{2^n},\ \frac{1}{2^n}+\frac{1}{2^{(n-1)^2}}-\frac{1}{2^{n^2}}\right),\\ \{0\} & \text{otherwise}. \end{cases}$$

Hence

$$(\partial f)^{-1}(0)=\mathbb{R}\setminus\bigcup_{n\in\mathbb{N}_1}\left(\frac{1}{2^n},\ \frac{1}{2^n}+\frac{1}{2^{(n-1)^2}}-\frac{1}{2^{n^2}}\right)$$

and

$$d(0,\partial f(x))=1\quad\forall x\notin(\partial f)^{-1}(0).$$

This implies that

$$d(x,(\partial f)^{-1}(0))\le d(0,\partial f(x))^p\quad\forall(x,p)\in(-1,\ 1)\times(0,\ +\infty),$$

and so ∂f is p-order metrically subregular at 0 for 0.

In terms of Hölder generalized directional derivative, the following proposition provides necessary and/or sufficient conditions for Hölder sharp minimizers.

PROPOSITION 4.2. *Let X be a Banach space and $f:X\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x}\in\mathrm{dom}(f)$ and $q\in(1,\ +\infty)$. Consider the following statements:*

(i) *$\bar{x}$ is a q-order sharp minimizer of f.*

(ii) *$0\in\partial f(\bar{x})$ and the q-order generalized directional derivative $d^q f(\bar{x},0)$ is strictly positive.*

Then (i)⇒(ii). *If, in addition, X is finite dimensional, then* (i)⇔(ii).

Proof. Suppose that there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa\|x-\bar{x}\|^q \le f(x) - f(\bar{x}) \quad \forall x \in B(\bar{x}, \delta).$$

Then, $0 \in \partial f(\bar{x})$ and for any $h \in X$ there exists $\delta' > 0$ such that

$$\kappa t^q \|h'\|^q \le f(\bar{x}+th') - f(\bar{x}) \quad \forall (t, h') \in (0, \delta') \times B(h, \delta').$$

It follows that $\kappa\|h\|^q \le d^q f(\bar{x}, 0)(h)$ for all $h \in X$. Therefore $d^q f(\bar{x}, 0)$ is strictly positive. Next suppose that X is finite dimensional and that $0 \in \partial f(\bar{x})$ and $d^q f(\bar{x}, 0)$ is strictly positive. We need to show that $\bar{x}$ is a q-order sharp minimizer of f. To do this, suppose to the contrary that there exists a sequence $\{x_n\}$ convergent to $\bar{x}$ such that $\frac{\|x_n-\bar{x}\|^q}{n} > f(x_n) - f(\bar{x})$ for all $n \in \mathbb{N}$. For each n, let $h_n := \frac{x_n - \bar{x}}{\|x_n-\bar{x}\|}$. We assume without loss of generality that $h_n \to h$ (taking a subsequence if necessary). Then $\|h\| = 1$, $t_n := \|x_n - \bar{x}\| \to 0$, and $d^q f(\bar{x}, 0)(h) \le \liminf_{n\to\infty} \frac{f(\bar{x}+t_n h_n) - f(\bar{x})}{t_n^q} \le 0$, contradicting the strict positivity of $d^q f(\bar{x}, 0)$. The proof is complete. ∎

Along the same line as in Theorem 4.1, the following result describes the Hölder stable sharp minimizers in terms of the Hölder strong metric regularity of the corresponding subdifferential mapping.

THEOREM 4.3. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ be a local minimizer of f and p be a positive number. Consider the following statements:*

(i) *∂f is p-order strongly metrically regular at $\bar{x}$ for 0.*

(ii) *$\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f.*

Then (i)⇒(ii). *Moreover, if f is convex, then* (i)⇔(ii).

Proof. First we prove (i)⇒(ii). To do this, suppose that (i) holds. By Lemma 2.6, there exist $\delta, \kappa \in (0, +\infty)$ and a mapping $u^* \mapsto x_{u^*}$ from $B_{X^*}(0, \delta)$ to X with $x_0 = \bar{x}$ such that $x_{u^*} \in (\partial f)^{-1}(u^*)$ and

$$\kappa\|u - x_{u^*}\| \le d(u^*, \partial f(u))^p \quad \forall (u, u^*) \in B_X(\bar{x}, \delta) \times B_{X^*}(0, \delta). \tag{4.8}$$

Setting $u = \bar{x}$ and noting that $0 \in \partial f(\bar{x})$, it follows that

$$\kappa\|\bar{x} - x_{u^*}\| \le \|u^*\|^p \quad \forall u^* \in B_{X^*}(0, \delta) \tag{4.9}$$

and so

$$\lim_{u^*\to 0} \|\bar{x} - x_{u^*}\| = 0. \tag{4.10}$$

On the other hand, setting $u^* = 0$, (4.8) implies that

$$\kappa\|x - \bar{x}\| \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta).$$

Since $\bar{x}$ is a local minimizer of f, there exists $r > 0$ such that (4.1) holds. Hence, by Theorem 4.1(i), we have that (4.3) holds with τ and η defined as in (4.4). Noting that $\partial f_{u^*}(x) = \partial f(x) - u^*$ for all $(u^*, x) \in X^* \times X$, (4.8) means that

$$\kappa\|u - x_{u^*}\| \le d(0, \partial f_{u^*}(u))^p \quad \forall (u, u^*) \in B_X(\bar{x}, \delta) \times B_{X^*}(0, \delta), \tag{4.11}$$

where f_{u^*} is as in (1.5). Let $\delta' := \min\{\frac{\eta}{2}, \left(\frac{\kappa\eta}{2}\right)^{\frac{1}{p}}, \tau\left(\frac{\eta}{2}\right)^{\frac{1}{p}}\}$; by (4.10), (4.11), and Theorem 4.1(i) (applied to f_{u^*} and x_{u^*} in place of f and $\bar{x}$), it suffices to show that

$$f_{u^*}(x_{u^*}) = \min_{x \in B_X[x_{u^*}, \frac{\eta}{2}]} f_{u^*}(x) \qquad \forall u^* \in B_{X^*}(0, \delta'). \tag{4.12}$$

Let $u^* \in B_{X^*}(0, \delta')$. Then, by (4.9) and the definition of δ', one has

$$\|\bar{x} - x_{u^*}\| \le \frac{\|u^*\|^p}{\kappa} < \frac{\delta'^p}{\kappa} \le \frac{\eta}{2}.$$

It follows that there exists $\eta' \in (\frac{\eta}{2},\ \eta)$ such that $B_X[x_{u^*}, \frac{\eta}{2}] \subset B_X[\bar{x}, \eta']$. Thus, to prove (4.12), we only need to show that

$$f_{u^*}(x_{u^*}) = \min_{x \in B_X[\bar{x}, \eta']} f_{u^*}(x). \tag{4.13}$$

Let $\{\varepsilon_n\}$ be a sequence in $(0,\ +\infty)$ convergent to 0 and take a sequence $\{x_n\}$ in $B_X[\bar{x}, \eta']$ such that

$$f_{u^*}(x_n) < \inf_{x \in B_X[\bar{x}, \eta']} f_{u^*}(x) + \varepsilon_n^2 \quad \forall n \in \mathbb{N}. \tag{4.14}$$

By the Ekeland variational principle, for each $n \in \mathbb{N}$ there exists $u_n \in B_X[\bar{x}, \eta']$ such that

$$\|u_n - x_n\| < \varepsilon_n \tag{4.15}$$

and

$$f_{u^*}(u_n) \le f_{u^*}(x) + \varepsilon_n \|x - u_n\| \quad \forall x \in B_X[\bar{x}, \eta'], \tag{4.16}$$

and so

$$f(u_n) - f(\bar{x}) \le \langle u^*, u_n - \bar{x}\rangle + \varepsilon_n \|u_n - \bar{x}\| \le (\delta' + \varepsilon_n)\|u_n - \bar{x}\| \le \left(\tau\left(\frac{\eta}{2}\right)^{\frac{1}{p}} + \varepsilon_n\right)\|u_n - \bar{x}\|.$$

Noting that $u_n \in B_X[\bar{x}, \eta'] \subset B_X(\bar{x}, \eta)$, it follows from (4.3) that

$$\tau\|u_n - \bar{x}\|^{\frac{1}{p}} \le \tau\left(\frac{\eta}{2}\right)^{\frac{1}{p}} + \varepsilon_n.$$

Hence $\limsup_{n\to\infty} \tau\|u_n - \bar{x}\|^{\frac{1}{p}} \le \tau\left(\frac{\eta}{2}\right)^{\frac{1}{p}} < \tau\eta'^{\frac{1}{p}}$ and so $\|u_n - \bar{x}\| < \eta'$ for all sufficiently large $n \in \mathbb{N}$. This and (4.16) imply that

$$0 \in \partial f_{u^*}(u_n) + \varepsilon_n B_{X^*}$$

and so there exists $u_n^* \in B_{X^*}$ such that $\varepsilon_n u_n^* \in \partial f_{u^*}(u_n)$. Thus, by (4.11), one has

$$\kappa\|u_n - x_{u^*}\| \le d(0, \partial f_{u^*}(u_n))^p \le \|\varepsilon_n u_n^*\|^p \to 0.$$

It follows from (4.15) that $x_n \to x_{u^*}$. Since f is lower semicontinuous, this and (4.14) imply that (4.13) holds.

Now consider the case that f is convex. It suffices to show that (ii)⇒(i). To do this, there exist $\delta_0, r_0, \tau_0 \in (0,\ +\infty)$ such that for each $u^* \in B_{X^*}(0, \delta_0)$ there exists $x_{u^*} \in B_X(\bar{x}, r_0)$, with $x_0 = \bar{x}$, satisfying the following property:

$$\tau_0\|x - x_{u^*}\|^{\frac{1+p}{p}} \le f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X(\bar{x}, r_0), \tag{4.17}$$

where f_{u^*} is as in (1.5). Hence

$$\tau_0\|x - \bar{x}\|^{\frac{1+p}{p}} \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, r_0)$$

and

$$\tau_0\|\bar{x} - x_{u^*}\|^{\frac{1+p}{p}} \le f_{u^*}(\bar{x}) - f_{u^*}(x_{u^*}) \quad \forall u^* \in B_{X^*}(0, \delta_0).$$

It follows that

$$\begin{aligned} f_{u^*}(x_{u^*}) &= f(x_{u^*}) - \langle u^*, x_{u^*}\rangle \ge f(\bar{x}) - \langle u^*, x_{u^*}\rangle \\ &= f_{u^*}(\bar{x}) - \langle u^*, x_{u^*} - \bar{x}\rangle \ge f_{u^*}(\bar{x}) - \|u^*\| r_0 \end{aligned}$$

and so

$$\tau_0\|\bar{x} - x_{u^*}\|^{\frac{1+p}{p}} \le \|u^*\| r_0 \quad \forall u^* \in B_{X^*}(0, \delta_0).$$

Letting $\delta := \min\{\delta_0, \frac{\tau_0 r_0^{\frac{1}{p}}}{2^{\frac{1+p}{p}}}\}$, this implies that $\|\bar{x} - x_{u^*}\| < \frac{r_0}{2}$ for all $u^* \in B_{X^*}(0, \delta)$. Hence

$$B_X\Big(x_{u^*}, \frac{r_0}{2}\Big) \subset B_X(\bar{x}, r_0) \quad \forall u^* \in B_{X^*}(0, \delta).$$

It follows from (4.17) that

$$\tau_0\|x - x_{u^*}\|^{\frac{1+p}{p}} \le f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X\Big(x_{u^*}, \frac{r_0}{2}\Big) \text{ and } \forall u^* \in B_{X^*}(0, \delta).$$

Thus, by Theorem 4.1(ii) (applied to f_{u^*} and x_{u^*} replacing respectively f and $\bar{x}$), one has

$$\tau_0^p\|x - x_{u^*}\| \le d(0, \partial f_{u^*}(x))^p \quad \forall x \in B_X(x_{u^*}, \delta) \text{ and } u^* \in B_{X^*}(0, \delta).$$

Noting that $d(0, \partial f_{u^*}(x)) = d(u^*, \partial f(x))$ and $\lim_{u^*\to 0} x_{u^*} = \bar{x}$, this and Lemma 2.6 imply that ∂f is p-order strong metrically regular at $\bar{x}$ for 0. The proof is complete. ▯

5. Hölder tilt-stable minima and Hölder stable minima. This section mainly *considers, for a proper lower semicontinuous function f, the relationship between the following two properties of a point $\bar{x} \in \mathrm{dom}(f)$:* (i) *$\bar{x}$ is a stable q-order sharp minimizer of f and* (ii) *$\bar{x}$ is a tilt-stable p-order minimizer of f.*

Proposition 5.1. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ and $p \in (0, +\infty)$. Suppose that there exist $r, \delta, L \in (0, +\infty)$ and $M : B_{X^*}(0, \delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that* (1.7) *holds and*

$$\arg\min_{x\in B_X[\bar{x},r]} f_{u^*}(x) = \{M(u^*)\} \quad \forall u^* \in B_{X^*}(0, \delta). \tag{5.1}$$

Then the conjugate function $(f + \delta_{B_X[\bar{x},r]})^*$ is $C^{1,p}$-smooth and

$$\nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*) = M(u^*) \quad \forall u^* \in B_{X^*}(0, \delta). \tag{5.2}$$

Consequently,

$$\|\nabla(f + \delta_{B_X[\bar{x},r]})^*(x^*) - \nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)\| \le L\|x^* - u^*\|^p \tag{5.3}$$

for all $x^*, u^* \in B_{X^*}(0, \delta)$.

Proof. By (5.1), one has

$$(f+\delta_{B_X[0,r]})^*(u^*)=\langle u^*, M(u^*)\rangle - f(M(u^*))\quad \forall u^*\in B_{X^*}(0,\delta), \tag{5.4}$$

and hence $M(u^*)\in\partial(f+\delta_{B_X[\bar{x},r]})^*(u^*)$ for all $u^*\in B_{X^*}(0,\delta)$. Thus, M is a norm-norm continuous selection of $\partial(f+\delta_{B_X[\bar{x},r]})^*$ on $B_{X^*}(0,\delta)$. Since $(f+\delta_{B_X[\bar{x},r]})^*$ is a proper lower semicontinuous convex function on X^*, it follows from [27] that $(f+\delta_{B_X[\bar{x},r]})^*$ is Fréchet differentiable on $B_{X^*}(0,\delta)$ and

$$\nabla(f+\delta_{B_X[\bar{x},r]})^*(u^*)=M(u^*)\quad \forall u^*\in B_{X^*}(0,\delta).$$

The proof is complete. ∎

Clearly, (1.6) means $M(u^*)\in\arg\min_{x\in B_X[\bar{x},r]} f_{u^*}(x)$ for all $u^*\in B_{X^*}(0,\delta)$, which is formally weaker than (5.1).

PROPOSITION 5.2. *Let X be a Banach space and $f: X\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x}\in\operatorname{dom}(f)$ and let p, r, δ, L be positive constants. Let $M: B_{X^*}(0,\delta)\to B_X[\bar{x},r]$ be a mapping with $M(0)=\bar{x}$ such that* (1.7) *holds. Then,* (5.1) *holds if and only if*

$$\operatorname*{arg\,min}_{x\in B_X[\bar{x},r]}\left((f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x)-\langle u^*, x\rangle\right)=\{M(u^*)\}\quad \forall u^*\in B_{X^*}(0,\delta) \tag{5.5}$$

and

$$f(x)=(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x)\quad \forall x\in\left(\partial(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}\right)^{-1}(B_{X^*}(0,\delta)). \tag{5.6}$$

Consequently, $\bar{x}$ is a tilt-stable p-order minimizer of f if and only if there exist $r,\delta\in(0,+\infty)$ such that (5.6) *holds and $\bar{x}$ is a tilt-stable p-order minimizer of the convex function $(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}$.*

Proof. For the sufficiency part, suppose that (5.5) and (5.6) hold. Since

$$\operatorname{dom}((f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}})\subset B_X[\bar{x},r],$$

(5.5) implies that

$$0\in\partial\left((f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}-u^*\right)(M(u^*))\quad \forall u^*\in B_{X^*}(0,\delta),$$

that is,

$$M(u^*)\in\left(\partial(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}\right)^{-1}(u^*)\quad \forall u^*\in B_{X^*}(0,\delta).$$

This and (5.6) imply that

$$f(M(u^*))=(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*))\quad \forall u^*\in B_{X^*}(0,\delta).$$

Noting that

$$f(x)\geq(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x)\quad \forall x\in B_X[\bar{x},r], \tag{5.7}$$

it follows from (5.5) that

$$\operatorname*{arg\,min}_{x\in B_X[\bar{x},r]} f_{u^*}=\operatorname*{arg\,min}_{x\in B_X[\bar{x},r]}(f(x)-\langle u^*, x\rangle)=\{M(u^*)\}\quad \forall u^*\in B_{X^*}(0,\delta).$$

Therefore, (5.1) holds. This shows the sufficiency part.

To prove the necessity part, suppose that (5.1) holds. Then, for all $u^*\in B_{X^*}(0,\delta)$,

$$\langle u^*, M(u^*)\rangle - f(M(u^*)) = (f+\delta_{B_X[\bar{x},r]})^*(u^*) = (f+\delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}}(u^*),$$

and so

$$f(M(u^*)) = \langle u^*, M(u^*)\rangle - (f+\delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}}(u^*) \le (f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*)).$$

It follows from (5.7) that

$$f(M(u^*)) = (f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*)) \quad \forall u^*\in B_{X^*}(0,\delta). \tag{5.8}$$

By Proposition 5.1, (1.7) and (5.1) imply that

$$\partial(f+\delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}}(u^*) = \{M(u^*)\} \quad \forall u^*\in B_{X^*}(0,\delta).$$

Noting that

$$u^*\in\partial(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x) \Leftrightarrow x\in\partial(f+\delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}}(u^*),$$

it follows from (5.8) that (5.6) holds. Since $(f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}$ is a lower semicontinuous convex function,

$$\arg\min\left((f+\delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}} - u^*\right) = X\cap\partial(f+\delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}}(u^*) = \{M(u^*)\}$$

for all $u^*\in B_{X^*}(0,\delta)$. This means that (5.5) holds. The proof is complete. ∎

THEOREM 5.3. *Let X be a Banach space and $f: X\to\overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $p,\kappa,\tau\in(0,\ +\infty)$ and $\bar{x}\in\mathrm{dom}(f)$. Then the following statements hold:*

(i) *If $\bar{x}$ is a tilt-stable p-order minimizer of f with modulus κ, then $\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f with modulus $\frac{p}{(1+p)\kappa^{\frac{1}{p}}}$.*

(ii) *If $\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f with modulus τ, then $\bar{x}$ is a tilt-stable p-order minimizer of f with modulus $\frac{1}{(2\tau)^p}$.*

Consequently, $\bar{x}$ is a tilt-stable p-order minimizer of f if and only if $\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f.

Proof. To prove (i), suppose that there exist $r,\delta\in(0,\ +\infty)$ and $M: B_{X^*}(0,\delta)\to B_X[\bar{x},r]$ such that (1.7) and (5.1) hold. It follows from Proposition 5.1 that (5.2) and (5.3) hold. This, together with Proposition 3.1 (applied to $g=(f+\delta_{B_X[\bar{x},r]})^*$), implies that there exist $r'\in(0,\ r)$ and $\delta'\in(0,\ \delta)$ such that

$$(f+\delta_{B_X[\bar{x},r]})^{**}(x) \ge \langle x,u^*\rangle - (f+\delta_{B_X[\bar{x},r]})^*(u^*) + \frac{p}{(1+p)\kappa^{\frac{1}{p}}}\|x-M(u^*)\|^{\frac{1+p}{p}}$$

for all $x\in B_X(\bar{x},r')$ and $u^*\in B_{X^*}(0,\delta')$. Noting that

$$(f+\delta_{B_X[\bar{x},r]})^{**}(x) \le (f+\delta_{B_X[\bar{x},r]})(x) = f(x) \quad \forall x\in B(\bar{x},r'),$$

it follows that

$$f_{u^*}(x) + (f+\delta_{B_X[\bar{x},r]})^*(u^*) \ge \frac{p}{(1+p)\kappa^{\frac{1}{p}}}\|x-M(u^*)\|^{\frac{1+p}{p}}$$

for all $x\in B_X(\bar{x},r')$ and $u^*\in B_{X^*}(0,\delta')$. Noting (by (5.1)) that

$$(f+\delta_{B_X[\bar{x},r]})^*(u^*) = \langle u^*, M(u^*)\rangle - f(M(u^*)) = -f_{u^*}(M(u^*)) \quad \forall u^*\in B_{X^*}(0,\delta),$$

this shows that $\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f with modulus $\frac{p}{(1+p)\kappa^{\frac{1}{p}}}$.

For (ii), suppose that there exist $\delta, r \in (0,\ +\infty)$ such that for any $u^* \in B_{X^*}(0,\delta)$ there exists $x_{u^*} \in B_X[\bar{x}, r]$ satisfying

$$\tau\|x - x_{u^*}\|^{\frac{1+p}{p}} \le f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X[\bar{x}, r]. \tag{5.9}$$

Let $u_1^*, u_2^* \in B_{X^*}(0,\delta)$. Then

$$\begin{aligned} 2\tau\|x_{u_2^*} - x_{u_1^*}\|^{\frac{1+p}{p}} &\le f_{u_1^*}(x_{u_2^*}) - f_{u_1^*}(x_{u_1^*}) + f_{u_2^*}(x_{u_1^*}) - f_{u_2^*}(x_{u_2^*}) \\ &= \langle u_1^* - u_2^*, x_{u_1^*} - x_{u_2^*}\rangle \\ &\le \|u_1^* - u_2^*\|\|x_{u_1^*} - x_{u_2^*}\| \end{aligned}$$

and so

$$\|x_{u_2^*} - x_{u_1^*}\| \le \frac{1}{(2\tau)^p}\|u_1^* - u_2^*\|^p.$$

Noting (by (5.9)) that

$$\operatorname*{arg\,min}_{x \in B_X[\bar{x},r]} f_{u^*}(x) = \{x_{u^*}\} \quad \forall u^* \in B_{X^*}(0,\delta),$$

this shows that $\bar{x}$ gives a tilt-stable p-order minimum of f with modulus $\frac{1}{(2\tau)^p}$. Therefore (ii) holds. The proof is complete. ∎

COROLLARY 5.4. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \operatorname{dom}(f)$. Then the following statements are equivalent:*

(i) *There exists $p \in (1,\ +\infty)$ such that $\bar{x}$ is a tilt-stable p-order minimizer of f.*

(ii) *There exist $r, \delta \in (0,\ +\infty)$ such that*

$$\operatorname*{arg\,min}_{x \in B_X[\bar{x},r]} f_{u^*} = \{\bar{x}\} \quad \forall u^* \in B_{X^*}(0,\delta). \tag{5.10}$$

(iii) *For any $p \in (0,\ +\infty)$, $\bar{x}$ is a tilt-stable p-order minimizer of f.*

(iv) *There exists $q \in (1,\ 2)$ such that $\bar{x}$ is a stable q-order sharp minimizer of f.*

(v) *For any $q \in (1,\ +\infty)$, there exist $\tau, r, \delta \in (0,\ +\infty)$ such that*

$$\tau\|x - \bar{x}\|^q \le f_{u^*}(x) - f_{u^*}(\bar{x}) \quad \forall (x, u^*) \in B_X[\bar{x}, r] \times B_{X^*}(0,\delta)$$

(in particular $\bar{x}$ is a stable q-order sharp minimizer of f).

Proof. (ii)⇒(iii)⇒(i) and (v)⇒(iv) are trivial, while (i)⇔(iv) is immediate from Theorem 5.1. For (i)⇒(ii), suppose there exists $p \in (1,\ +\infty)$ such that $\bar{x}$ is a tilt-stable p-order minimizer of f. Then there exist $r, \delta, L \in (0,\ +\infty)$ and $M : B_{X^*}(0,\delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that (1.7) and (5.1) hold. Since $p > 1$, it is easy to verify from (1.7) that M is differentiable on $B_{X^*}(0,\delta)$ and $\nabla M(u^*) = 0$ for all $u^* \in B_{X^*}(0,\delta)$. Hence $M(u^*) = M(0) = \bar{x}$ for all $u^* \in B_{X^*}(0,\delta)$. It follows from (5.1) that (5.10) holds, hence (i)⇒(ii). Thus (i)–(iv) are equivalent. Further, from (5.10) and Theorem 5.1(i) as well as its proof, one can show that (iii)⇒(v). The proof is complete. ∎

The following theorem for convex functions is immediate from Theorems 4.3 and 5.3 and Proposition 3.2.

THEOREM 5.5. *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function. Let $\bar{x} \in \operatorname{dom}(f)$ be a minimizer of f and p be a positive number. Then the following statements are equivalent:*

(i) $\bar{x}$ *is a tilt-stable p-order minimizer of f.*
(ii) $\bar{x}$ *is a stable* $\frac{1+p}{p}$*-order sharp minimizer of f.*
(iii) ∂f *is p-order strongly metrically regular at* $\bar{x}$ *for* 0*.*
(iv) f^* *is* $C^{1,p}$*-smooth on a neighborhood of* 0*.*

Given $r, \delta \in (0,\ +\infty)$, note that, trivially,

$$(5.10) \Rightarrow B_{X^*}(0, \delta) \subset \partial f(\bar{x});$$

if f is convex, then the following equivalences hold:

$$(5.10) \Leftrightarrow \arg\min_{x \in X} f_{u^*} = \{\bar{x}\} \quad \forall u^* \in B_{X^*}(0, \delta)$$

and

$$\begin{aligned} B_{X^*}(0,\delta) &\subset \partial f(\bar{x}) \\ &\Leftrightarrow B_{X^*}(0, \delta - \|u^*\|) \subset \partial f_{u^*}(\bar{x}) \ \ \forall u^* \in B_{X^*}(0,\delta) \\ &\Leftrightarrow (\delta - \|u^*\|)\|x - \bar{x}\| \le f_{u^*}(x) - f_{u^*}(\bar{x}) \ \ \forall (u^*, x) \in B_{X^*}(0,\delta) \times X \\ &\Leftrightarrow (5.10). \end{aligned}$$

On the other hand, $\bar{x}$ is a 1-order sharp minimizer of f if and only if there exist $\kappa, \delta \in (0,\ +\infty)$ such that (1.3) holds with $q = 1$, while (1.1) with $q = 1$ implies that $B_{X^*}(0, \kappa) \subset \partial f(\bar{x})$. Thus, by Theorem 5.5 and Corollary 5.4, we have the following corollary.

COROLLARY 5.6. *Let X be a Banach space and* $f : X \to \overline{\mathbb{R}}$ *be a proper lower semicontinuous convex function. Let* $(\bar{x}, 0) \in \mathrm{gph}(\partial f)$*. Then the following statements are equivalent:*

(i) $\bar{x}$ *is a* 1*-order sharp minimizer of f.*
(ii) *For any* $q \in [1, 2)$*, there exist* $\kappa, \delta, r > 0$ *such that*

$$\kappa \|x - \bar{x}\|^q \le f_{u^*}(x) - f_{u^*}(\bar{x}) \quad \forall (x, u^*) \in B_X(\bar{x}, r) \times B_{X^*}(0, \delta).$$

(iii) *There exists* $\delta > 0$ *such that* $B_{X^*}(0, \delta) \subset \partial f(\bar{x})$*.*
(iv) *There exists* $\delta > 0$ *such that* $(\partial f)^{-1}(u^*) = \{\bar{x}\}$ *for all* $u^* \in B_{X^*}(0, \delta)$*.*
(v) *There exists* $p_0 \in (1,\ +\infty)$ *such that* ∂f *is* p_0*-order strongly metrically regular at* $\bar{x}$ *for* 0*.*
(vi) *For any* $p \in (1,\ +\infty)$*,* ∂f *is p-order strongly metrically regular at* $\bar{x}$ *for* 0*.*
(vii) *There exists* $\delta > 0$ *such that*

$$f^*(x^*) = \langle x^*, \bar{x} \rangle - f(\bar{x}) \quad \forall x^* \in B_{X^*}(\bar{x}^*, \delta).$$

Under the assumption that X is a Hilbert space, Theorem 5.5 can be extended to a certain nonconvex case.

THEOREM 5.7. *Let X be a Hilbert space and* $f : X \to \overline{\mathbb{R}}$ *be a proper lower semicontinuous function. Let* $\bar{x} \in \mathrm{dom}(f)$ *be a minimizer of f and p be a positive number. Suppose that f is prox-regular and subdifferentially continuous at* $\bar{x}$ *for* 0*. Then the following statements are equivalent:*

(i) $\bar{x}$ *is a tilt-stable p-order minimizer of f.*
(ii) $\bar{x}$ *is a stable* $\frac{1+p}{p}$*-order sharp minimizer of f.*
(iii) ∂f *is p-order strongly metrically regular at* $\bar{x}$ *for* 0*.*

(iv) *There exist* $r, \delta \in (0, +\infty)$ *such that* $(f + \delta_{B_X[\bar{x},r]})^*$ *is* $C^{1,p}$*-smooth on* $B_{X^*}(0, \delta)$ *and*

$$\langle u^*, \nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)\rangle - f(\nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)) = (f + \delta_{B_X[\bar{x},r]})^*(u^*) \tag{5.11}$$

for all $u^* \in B_{X^*}(0, \delta)$.

Proof. (iii)⇒(ii)⇔(i) are immediate from Theorems 4.3 and 5.3. Now suppose that (i) holds. Then there exist $\delta, r, L \in (0, +\infty)$ and $M : B_{X^*}(0, \delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that (1.7) and (5.1) hold. Thus, by Theorem 5.3 and Propositions 5.1 and 5.2, we have that $(f + \delta_{B_X[\bar{x},r]})^*$ is $C^{1,p}$-smooth on $B_{X^*}(0, \delta)$ and (5.6) holds. Let $h := (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}$. Then, $h^* = (f + \delta_{B_X[\bar{x},r]})^*$ and so h^* is $C^{1,p}$-smooth on $B_{X^*}(0, \delta)$. By the Fenchel duality formula, we have, for all $u^* \in B_{X^*}(0, \delta)$ and $x := \nabla h^*(u^*)$, that $u^* \in \partial h(x)$ (so $(f + \delta_{B_X[\bar{x},r]})(x) = h(x)$ by (5.6)) and

$$\langle u^*, x\rangle - h(x) = h^*(u^*),$$

that is, (5.11) holds. Hence (i)⇒(iv) holds. Thus, it suffices to show that (iv)⇒(iii) holds. To do this, suppose that there exist $r, \delta \in (0, +\infty)$ such that $(f + \delta_{B_X[\bar{x},r]})^*$ is $C^{1,p}$-smooth on $B_{X^*}(0, \delta)$ and (5.11) holds. Let $M : B_{X^*}(0, \delta) \to X$ be such that

$$M(u^*) := \nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*) \quad \forall u^* \in B_{X^*}(0, \delta).$$

Noting that $(f + \delta_{B_X[\bar{x},r]})^*_{\overline{\mathrm{co}}} = (f + \delta_{B_X[\bar{x},r]})^*$, it follows that

$$\langle u^*, M(u^*)\rangle - (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*)) = (f + \delta_{B_X[\bar{x},r]})^*(u^*)$$

and

$$(\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}})^{-1}(u^*) = \partial(f + \delta_{B_X[\bar{x},r]})^*(u^*) = \{M(u^*)\}$$

for all $u^* \in B_{X^*}(0, \delta)$. Thus, by (5.11),

$$f(M(u^*)) = (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*)) \quad \forall u^* \in B_{X^*}(0, \delta) \tag{5.12}$$

and

$$\mathrm{gph}(\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}) \cap (X \times B_{X^*}(0, \delta)) = \{(u, u^*) : (u^*, u) \in \mathrm{gph}(M)\}. \tag{5.13}$$

By (5.13), $\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}$ is p-order strong metrically regular at $\bar{x}$ for 0 as M is the derivative of a $C^{1,p}$-smooth function on $B_{X^*}(0, \delta)$. To complete the proof for (iii), it suffices to show that there exist $\delta', r' \in (0, \infty)$ such that

$$\partial f(x) \cap B_{X^*}(0, \delta') = \partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x) \cap B_{X^*}(0, \delta') \quad \forall x \in B_X(\bar{x}, r'). \tag{5.14}$$

Take $\delta_0 \in (0, \delta)$ such that $M(B_{X^*}(0, \delta_0)) \subset B_X(\bar{x}, r)$. Thus, if $u^* \in B_{X^*}(0, \delta_0)$, then $M(u^*)$ lies in the open ball $B_X(\bar{x}, r)$ and, by (5.13) and (5.12), one has

$$\begin{aligned}\langle u^*, x - M(u^*)\rangle &\le (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x) - (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(M(u^*))\\ &= (f + \delta_{B_X[\bar{x},r]})_{\overline{\mathrm{co}}}(x) - f(M(u^*))\\ &\le f(x) - f(M(u^*))\end{aligned}$$

for all $x \in B_X[\bar{x}, r]$ and so $u^* \in \partial f(M(u^*))$. This and (5.13) imply that

$$\text{gph}(\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\text{co}}}) \cap (X \times B(0, \delta_0)) \subset \text{gph}(\partial f). \tag{5.15}$$

By Lemma 2.4 (thanks to the assumption that f is prox-regular and subdifferentially continuous at $\bar{x}$ for 0), there exist $\sigma, r_1, \delta_1 \in (0, +\infty)$ such that $\text{gph}(\partial f + \sigma I) \cap (B(\bar{x}, r_1) \times B(\sigma\bar{x}, \delta_1))$ is a monotone set. Since $\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\text{co}}}$ is maximally monotone, $\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\text{co}}} + \sigma I$ is locally maximally monotone at $(\bar{x}, \sigma\bar{x})$ by Lemma 2.3. Thus, there exist $r', \delta' \in (0, +\infty)$ such that

$$r' < r_1, \quad \delta' + \sigma r' < \min\{\delta_0, \delta_1\} \tag{5.16}$$

and

$$A := \text{gph}\left(\partial(f + \delta_{B_X[\bar{x},r]})_{\overline{\text{co}}} + \sigma I\right) \cap (B(\bar{x}, r') \times B(\sigma\bar{x}, \delta'))$$

is a maximally monotone subset of $B(\bar{x}, r') \times B(\sigma\bar{x}, \delta')$. From (5.15) and (5.16), it is easy to verify that A is a subset of the monotone set $\text{gph}(\partial f + \sigma I) \cap (B(\bar{x}, r') \times B(\sigma\bar{x}, \delta'))$. Therefore, $A = \text{gph}(\partial f + \sigma I) \cap (B(\bar{x}, r') \times B(\sigma\bar{x}, \delta'))$. This shows that (5.14) holds. The proof is complete. ∎

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