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CALMNESS FOR L-SUBSMOOTH MULTIFUNCTIONS IN BANACH SPACES*

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Abstract. Using variational analysis techniques, we study subsmooth multifunctions in Banach spaces. In terms of the normal cones and coderivatives, we provide some characterizations for such multifunctions to be calm. Sharper results are obtained for Asplund spaces. We also present some exact formulas of the modulus of the calmness. As applications, we provide some error bound results on nonconvex inequalities, which improve and generalize the existing error bound results.

Key words. subsmoothness, calmness, metric subregularity, error bound, multifunction

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1. Introduction.

As an extension of convexity, prox-regularity of a set expresses a variational behavior of "order two" and plays an important role in variational analysis (see [5, 32, 34] and the references therein). Recently, Aussel, Daniilidis, and Thibault [1] considered a variational behavior of "order one" of a set and introduced subsmoothness, extending the notions of the smoothness and the prox-regularity. Motivated by their work, we consider (in section 3) a further weakened notion (called L-subsmooth).

The calmness property plays an important role in many issues in mathematical programming like exact penalty functions, optimality conditions, local error bounds, weak sharp minima, and so on. Recently, many authors studied calmness (cf. [8, 11, 9, 10, 17, 34, 43, 45] and the references therein). Let Y, X be Banach spaces and $M : Y \rightrightarrows X$ a multifunction. For $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$, recall that M is calm at $(\bar{y}, \bar{x})$ if there exist $\eta, \delta \in (0, +\infty)$ such that

$$d(x, M(\bar{y})) \leq \eta\|y - \bar{y}\| \quad \forall y \in B(\bar{y}, \delta) \text{ and } x \in M(y) \cap B(\bar{x}, \delta), \tag{1.1}$$

where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius r. Let $F(x) := \{y \in Y : x \in M(y)\}$ for all $x \in X$. As observed by Henrion and Outrata [10], the calmness of M at $(\bar{y}, \bar{x})$ is equivalent to the condition that there exist $\eta, \delta \in (0, +\infty)$ such that

$$d(x, F^{-1}(\bar{y}) \leq \eta d(\bar{y}, F(x)) \quad \forall x \in B(\bar{x}, \delta). \tag{1.2}$$

Following Dontchev and Rockafellar [6], (1.2) means that the generalized equation $\bar{y} \in F(x)$ is metrically subregular at $\bar{x}$. This property provides an estimate on how far a candidate x can be from the solution set of the generalized equation. A stronger property is the following: a multifunction F is said to be metrically regular at $\bar{x}$ for $\bar{y}$

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if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta)). \tag{1.3}$$

Both notions (of the metric regularity and the metric subregularity) have been studied by many authors (see [7, 6, 14, 20, 23, 24, 25, 28, 40, 43] and the references therein). In particular, it is well known (cf. [23, 24, 25, 34]) that if X and Y are finite-dimensional, then F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $D^*F(\bar{x}, \bar{y})^{-1}(0) = \{0\}$; moreover

$$\inf\{\tau > 0 : \text{ (1.3) holds}\} = \|D^*F(\bar{x}, \bar{y})^{-1}\|^- = \limsup_{(x,y)\xrightarrow{\mathrm{Gr}(F)}(\bar{x},\bar{y})} \|D^*F(x, y)^{-1}\|^-, \tag{1.4}$$

where $D^*F(x, y)$ is the coderivative of F at (x, y) and $\|D^*F(x, y)^{-1}\|^-$ denotes the inner norm of $D^*F(x, y)^{-1}$ (see section 2 for undefined terms and further notation). The modulus of the calmness of M at $(\bar{y}, \bar{x})$ is denoted by $\eta(M; \bar{y}, \bar{x})$ and defined by

$$\eta(M; \bar{y}, \bar{x}) := \inf\{\eta \in (0, \infty) : \text{ (1.1) holds for some } \delta \in (0, +\infty)\}. \tag{1.5}$$

The case $\eta(M; \bar{y}, \bar{x}) = \infty$ indicates that M is not calm at $(\bar{y}, \bar{x})$ (here and throughout we adopt the convention that the infimum over the empty set is ∞). In terms of the normal cone of $M(\bar{y})$, the derivative, or subdifferential, Henrion and Outrata [9], Henrion, Jourani, and Outrata [11], and Henrion and Jourani [8] gave sufficient conditions for $\eta(M; \bar{y}, \bar{x}) < +\infty$ in some special cases. Recently, in terms of the normal cone and coderivative, the authors [43] considered the case when M is a general closed convex multifunction between Banach spaces and provided some characterizations for $\eta(M; \bar{y}, \bar{x}) < +\infty$. This and (1.4) motivate us to seek some formulas for $\eta(M; \bar{y}, \bar{x})$ in terms of coderivative in the case when M is not necessarily convex. In section 4, for L-subsmooth multifunctions, we establish some such formulas and provide several sufficient and/or necessary conditions for the calmness. In section 5, as an application, we consider error bounds for inequalities. In particular, we extend some existing error bound results from the convex case to the nonconvex case.

2. Preliminaries. Let X be a Banach space. Let X^* and B_X denote the dual space and the closed unit ball of X, respectively.

For a closed subset A of X and $a \in A$, let $T_c(A, a)$ and $T(A, a)$ denote, respectively, the Clarke tangent cone and the contingent (Bouligand) cone of A at a; they are defined by

$$T_c(A, a) := \liminf_{x \xrightarrow{A} a, t \to 0^+} \frac{A - x}{t} \quad \text{and} \quad T(A, a) := \limsup_{t \to 0^+} \frac{A - a}{t},$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Thus, $v \in T_c(A, a)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all n, while $v \in T(A, a)$ if and only if there exist a sequence $\{v_n\}$ converging to v and a sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0 such that $a + t_n v_n \in A$ for all n. We denote by $N_c(A, a)$ the Clarke normal cone of A at a, that is,

$$N_c(A, a) := \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \ \ \forall h \in T_c(A, a)\}.$$

For $\varepsilon \geq 0$ and $a \in A$, the nonempty set

$$\hat{N}_\varepsilon(A, a) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x - a\rangle}{\|x - a\|} \leq \varepsilon \right\}$$

is called the set of Fréchet ε-normals of A at a. When $\varepsilon = 0$, $\hat{N}_\varepsilon(A, a)$ is a convex cone which is called the Fréchet normal cone of A at a and is denoted by $\hat{N}(A, a)$.

Let $N(A, a)$ denote the Mordukhovich normal cone (also known as the limiting or basic normal cone) of A at a, that is,

$$N(A, a) = \limsup_{x \xrightarrow{A} a, \varepsilon \to 0^+} \hat{N}_\varepsilon(A, x).$$

Thus, $x^* \in N(A, a)$ if and only if there exists a sequence $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (a, 0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \hat{N}_{\varepsilon_n}(A, x_n)$ for each $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. It is known that

$$\hat{N}(A, a) \subset N(A, a) \subset N_c(A, a) \tag{2.1}$$

(cf. [24, 25, 26]). It is known that if A is convex, then $T_c(A, a) = T(A, a)$ and

$$N_c(A, a) = \hat{N}(A, a) = \{x^* \in X^* : \langle x^*, x\rangle \leq \langle x^*, a\rangle \ \ \forall x \in A\}.$$

Recall that a Banach space X is called an Asplund space if every continuous convex function on X is Fréchet differentiable at each point of a dense subset of X (for other definitions and their equivalents, see [31, Definition 1.22 and Corollary 2.35]). It is well known (cf. [31]) that X is an Asplund space if and only if every separable subspace of X has a separable dual space. In the case when X is an Asplund space, Mordukhovich and Shao [26] proved that

$$N_c(A, a) = \mathrm{cl}^*(\mathrm{co}(N(A, a))) \ \text{ and } \ N(A, a) = \limsup_{x \xrightarrow{A} a} \hat{N}(A, x).$$

The following approximate projection result (recently established in [44]) will play an important role in the proofs of our main results.

LEMMA 2.1. *Let be A a nonempty closed subset of a Banach space X and let $\gamma \in (0,\ 1)$. Then for any $x \notin A$ there exist $a \in \mathrm{bd}(A)$ and $a^* \in N_c(A, a)$ with $\|a^*\| = 1$ such that*

$$\gamma\|x - a\| < \min\{d(x, A),\ \langle a^*, x - a\rangle\}.$$

If X is assumed to be an Asplund space, then above a^* can be chosen from $\hat{N}(A, a)$.

For a multifunction F between Banach spaces X and Y, the graph of F is defined by

$$\mathrm{Gr}(F) := \{(x, y) \in X \times Y : \ y \in F(x)\}.$$

As usual, F is said to be closed (resp., convex) if $\mathrm{Gr}(F)$ is a closed (resp., convex) subset of $X \times Y$. Let $(x, y) \in \mathrm{Gr}(F)$. The Clarke tangent and contingent derivatives $D_cF(x, y)$, $DF(x, y)$ of F at (x, y) are defined by

$$\mathrm{Gr}(D_cF(x, y)) = T_c(\mathrm{Gr}(F), (x, y)) \ \text{ and } \ \mathrm{Gr}(DF(x, y)) = T(\mathrm{Gr}(F), (x, y)),$$

respectively. Let $\hat{D}^*F(x, y)$, $D^*F(x, y)$, and $D_c^*F(x, y)$ denote the coderivatives of F at (x, y) associated, respectively, with the Fréchet, Mordukhovich, and Clarke normal structures; they are defined by

$$\hat{D}^*F(x, y)(y^*) := \{x^* \in X^* : \ (x^*, -y^*) \in \hat{N}(\mathrm{Gr}(F), (x, y))\} \quad \forall y^* \in Y^*,$$
$$D^*F(x, y)(y^*) := \{x^* \in X^* : \ (x^*, -y^*) \in N(\mathrm{Gr}(F), (x, y))\} \quad \forall y^* \in Y^*,$$

and

$$D_c^*F(x,y)(y^*) := \{x^* \in X^* :\ (x^*, -y^*) \in N_c(\mathrm{Gr}(F), (x,y))\} \quad \forall y^* \in Y^*.$$

The history of the coderivatives can be found in Mordukhovich's book [24, 25].

Let $G : X \rightrightarrows Y$ be a positively homogeneous multifunction (i.e., $\mathrm{Gr}(G)$ is a cone in $X \times Y$). Following Dontchev, Lewis, and Rockefeller [7], the inner norm of G is defined by $\|G\|^- := \sup_{x \in B_X} \inf_{y \in Gx} \|y\|$. For a cone K in X, let $\|G|_K\|^-$ be defined by $\|G|_K\|^- := \sup_{x \in B_X \cap K} \inf_{y \in Gx} \|y\|$. It is not difficult to verify that

$$\|G^{-1}|_C\|^- = \inf\{\tau > 0 :\ C \cap B_Y \subset \tau G(B_X)\}. \tag{2.2}$$

3. Subsmoothness of multifunctions. Throughout the remainder of this paper, X, Y, and Z denote Banach spaces. If additional conditions are imposed, they will be explicitly specified.

Let A be a subset of X and $a \in A$. Recall (see [5, 32, 34]) that A is prox-regular at a if there exist $\sigma, \delta \in (0,\ +\infty)$ such that

$$\langle x^* - u^*, x - u\rangle \geq -\sigma\|x - u\|^2$$

whenever $x, u \in B(a,\delta) \cap A$, $x^* \in N_c(A,x) \cap B_{X^*}$, and $u^* \in N_c(A,u) \cap B_{X^*}$. As an interesting extension of the prox-regularity, Aussel, Daniilidis, and Thibault [1] introduced and studied the following subsmoothness and semisubsmoothness: A is said to be

(a) subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^* - u^*, x - u\rangle \geq -\varepsilon\|x - u\|$$

whenever $x, u \in B(a,\delta) \cap A$, $x^* \in N_c(A,x) \cap B_{X^*}$, and $u^* \in N_c(A,u) \cap B_{X^*}$;

(b) semisubsmooth at $a \in A$ if

$$\langle x^* - a^*, x - a\rangle \geq -\varepsilon\|x - a\|$$

whenever $x \in B(a,\delta) \cap A$, $x^* \in N_c(A,x) \cap B_{X^*}$, and $a^* \in N_c(A,a) \cap B_{X^*}$.

It is easy to verify that A is subsmooth at $a \in A$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - u\rangle \leq \varepsilon\|x - u\|$$

whenever $x, u \in B(a,\delta) \cap A$ and $u^* \in N_c(A,u) \cap B_{X^*}$. In the above (b), setting $x^* = 0$, one can define a weaker notion: A satisfies condition (S) at a if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle a^*, x - a\rangle \leq \varepsilon\|x - a\| \quad \forall x \in B(a,\delta) \cap A \ \text{ and } \ \forall a^* \in N_c(A,a) \cap B_{X^*}.$$

Clearly, if A satisfies condition (S), then $N_c(A,a) \subset \hat{N}(A,a)$ and so, by (2.1),

$$N_c(A,a) = N(A,a) = \hat{N}(A,a).$$

It is known (and easily verified) that

convexity⇒prox-regularity⇒subsmoothness⇒semisubsmoothness⇒condition (S).

In what follows, let $F : X \rightrightarrows Y$ be a closed multifunction, and let $a \in X$ and $b \in F(a)$.

DEFINITION 3.1. *We say that F is subsmooth (resp., satisfies condition* (S)*) at (a,b) if $\mathrm{Gr}(F)$ is subsmooth (resp., satisfies condition* (S)*) at (a,b).*

Now we introduce a few new notions which are weaker than the subsmoothness but stronger than condition (S). They will play an important role in our analysis.

DEFINITION 3.2. *We say that*

(i) *F is L-subsmooth at (a,b) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle u^*, x-a\rangle + \langle v^*, y-v\rangle \le \varepsilon(\|x-a\| + \|y-v\|) \tag{3.1}$$

whenever $v \in F(a)\cap B(b,\delta)$, $(u^*,v^*) \in N_c(\mathrm{Gr}(F),(a,v))\cap(B_{X^*}\times B_{Y^*})$, and $(x,y) \in \mathrm{Gr}(F)$ with $\|x-a\|+\|y-b\| < \delta$;

(i′) *F is $\mathcal{L}$-subsmooth at (a,b) if F^{-1} is L-subsmooth at (b,a);*

(ii) *F is weakly L-subsmooth if same as in* (i) *but the Clarke normal cone $N_c(\mathrm{Gr}(F),\cdot)$ is replaced with the Mordukhovich normal cone $N(\mathrm{Gr}(F),\cdot)$;*

(ii′) *F is weakly $\mathcal{L}$-subsmooth at (a,b) if F^{-1} is weakly L-subsmooth at (b,a).*

It is clear that the subsmoothness of F at (a,b) implies both the L-subsmoothness and the $\mathcal{L}$-subsmoothness of F at (a,b) and that the L-subsmoothness implies the weak L-subsmoothness.

Below we provide some sufficient conditions for subsmoothness of multifunctions.

PROPOSITION 3.3. *Suppose that F is defined by $F(x) = g(x)+\Omega$ for all $x \in X$, where $g : X \to Y$ is a smooth function and Ω is a closed subset of Y. Let $(a,b) \in \mathrm{Gr}(F)$. Then the following assertions hold.*

(i) *$T_c(\mathrm{Gr}(F),(a,b)) = \{(u,v) \in X\times Y :\ v \in g'(a)(u) + T_c(\Omega, b-g(a))\}$.*

(ii) *$N_c(\mathrm{Gr}(F),(a,b)) = \{(-(g'(a))^*(y^*), y^*) \in X^*\times Y^* :\ y^* \in N_c(\Omega, b-g(a))\}$.*

(iii) *If, in addition, Ω is subsmooth at $b-g(a)$, F is subsmooth at (a,b).*

Proof. (i) Let $(u,v) \in T_c(\mathrm{Gr}(F),(a,b))$ and take sequences $\omega_n \xrightarrow{\Omega} b-g(a)$ and $t_n \downarrow 0$. Then $(a, g(a)+\omega_n) \xrightarrow{\mathrm{Gr}(F)} (a,b)$ and hence there exists a sequence $\{(u_n,v_n)\}$ converging to (u,v) such that for all $n\in\mathbb{N}$,

$$(a, g(a)+\omega_n) + t_n(u_n,v_n) \in \mathrm{Gr}(F),$$

that is, $g(a)+\omega_n+t_nv_n \in g(a+t_nu_n)+\Omega$. This means that

$$\omega_n + t_n\left(v_n - \frac{g(a+t_nu_n)-g(a)}{t_n}\right) \in \Omega \quad \forall n \in \mathbb{N}.$$

Since $v_n - \frac{g(a+t_nu_n)-g(a)}{t_n} \to v - g'(a)(u)$, we obtain that $v-g'(a)(u) \in T_c(\Omega, b-g(a))$. This shows that the set on the left-hand side of (i) is contained in the set on the right-hand side. To prove the converse inclusion, let $u\in X$ and $v \in g'(a)(u)+T_c(\Omega,b-g(a))$; take arbitrary sequences $(x_n,y_n) \xrightarrow{\mathrm{Gr}(F)} (a,b)$ and $t_n\downarrow 0$. Then there exists a sequence $\{\omega_n\}$ in Ω such that $\omega_n = y_n - g(x_n) \to b-g(a)$, and so there exists a sequence $\{\tilde\omega_n\}$ in Ω such that $\frac{\tilde\omega_n-\omega_n}{t_n} \to v-g'(a)(u)$. By the smoothness of g, it follows that

$$v_n := \frac{g(x_n+t_nu)-g(x_n)}{t_n} + \frac{\tilde\omega_n-\omega_n}{t_n} \to v.$$

Note that, for each $n\in\mathbb{N}$,

$$y_n + t_nv_n = g(x_n)+\omega_n+t_nv_n = g(x_n+t_nu)+\tilde\omega_n \in F(x_n+t_nu),$$

that is, $(x_n, y_n) + t_n(u, v_n) \in \mathrm{Gr}(F)$. Therefore $(u, v) \in T_c(\mathrm{Gr}(F), (a, b))$. This shows that the converse inclusion holds.

(ii) This follows easily from (i).

(iii) Let $\varepsilon > 0$. Since g is smooth, there exist $M, r \in (0, +\infty)$ such that

$$\|g(x) - g(u)\| \leq M\|x - u\| \quad \forall x, u \in B(a, r) \tag{3.2}$$

and

$$\|g(x) - g(u) - g'(u)(x - u)\| \leq \frac{\varepsilon}{2}\|x - u\| \quad \forall x, u \in B(a, r). \tag{3.3}$$

By the subsmoothness of Ω at $b - g(a)$, there exists $\delta_1 > 0$ such that

$$\langle z^*, y - z\rangle \leq \frac{\varepsilon}{2(1 + M)}\|y - z\| \tag{3.4}$$

whenever $y, z \in \Omega \cap B(b - g(a), \delta_1)$ and $z^* \in N_c(\Omega, z) \cap B_{Y^*}$. By the continuity of g and the definition of F, there exists $\delta \in (0,\ r)$ such that

$$y - g(x) \in \Omega \cap B(b - g(a), \delta_1) \quad \forall (x, y) \in \mathrm{Gr}(F) \cap B((a, b), \delta). \tag{3.5}$$

Let $(x, y), (u, v) \in \mathrm{Gr}(F) \cap B((a, b), \delta)$ and $(u^*, v^*) \in N_c(\mathrm{Gr}(F), (u, v)) \cap (B_{X^*} \times B_{Y^*})$. Then, by (ii), $u^* = -(g'(u))^*(v^*)$ and $v^* \in N_c(\Omega, v - g(u)) \cap B_{Y^*}$. Thanks to (3.5), one can apply (3.4) to $y - g(x), v - g(u)$ in place of y, z and conclude that

$$\langle v^*, y - g(x) - (v - g(u))\rangle \leq \frac{\varepsilon}{2(1 + M)}\|y - g(x) - (v - g(u))\|;$$

it follows from (3.3) and (3.2) that

$$\begin{aligned}
\langle (u^*, v^*), (x, y) - (u, v)\rangle &= \langle v^*, -g'(u)(x - u) + y - v\rangle \\
&\leq \langle v^*, -(g(x) - g(u)) + y - v\rangle + \frac{\varepsilon\|x - u\|}{2} \\
&\leq \frac{\varepsilon}{2(1 + M)}\|y - v - g(x) + g(u)\| + \frac{\varepsilon}{2}\|x - u\| \\
&\leq \frac{\varepsilon}{2(1 + M)}\|y - v\| + \left(\frac{\varepsilon M}{2(M + 1)} + \frac{\varepsilon}{2}\right)\|x - u\| \\
&\leq \varepsilon(\|x - u\| + \|y - v\|).
\end{aligned}$$

This shows that F is subsmooth at (a, b). The proof is complete.

In the case when $g'(a)$ is surjective, Proposition 3.3 can be strengthened as follows.

PROPOSITION 3.4. *Suppose that $F : X \rightrightarrows Y$ is defined by $F(x) = G(g(x))$ for all $x \in X$, where $g : X \to Z$ is a smooth function and $G : Z \rightrightarrows Y$ is a closed multifunction. Let $(a, b) \in \mathrm{Gr}(F)$. Suppose that $g'(a)$ is surjective and that G is $\mathcal{L}$-subsmooth (resp., subsmooth) at $(g(a), b)$. Then F is $\mathcal{L}$-subsmooth (resp., subsmooth) at (a, b).*

To prove Proposition 3.4, we need the following lemma, which is of some independent interest.

LEMMA 3.5. *Let Θ be a closed subset of Y. Let $g : X \to Y$ be strictly differentiable at $\bar{x} \in g^{-1}(\Theta)$ and suppose that $g'(\bar{x})$ is surjective. Then*

$$T_c(g^{-1}(\Theta), \bar{x}) = (g'(\bar{x}))^{-1}(T_c(\Theta, g(\bar{x}))) \tag{3.6}$$

and

$$N_c(g^{-1}(\Theta),\bar{x}) = g'(\bar{x})^*(N_c(\Theta,g(\bar{x}))). \tag{3.7}$$

Proof. Let $h \in T_c(g^{-1}(\Theta),\bar{x})$ and take any sequences $y_n \xrightarrow{\Theta} g(\bar{x})$ and $t_n \downarrow 0$. By our assumptions on $\bar{x}$, the Lyusternik–Graves theorem (cf. [24, Theorem 1.57]) can be applied and so there exists $\mu \in (0,\ +\infty)$ such that for all large enough n,

$$d(\bar{x}, g^{-1}(y_n)) \le \mu\|g(\bar{x}) - y_n\|.$$

It follows that there exists $x_n \in g^{-1}(y_n) \subset g^{-1}(\Theta)$ such that $x_n \to \bar{x}$. Since $h \in T_c(g^{-1}(\Theta),\bar{x})$, there exists a sequence $h_n \to h$ such that $x_n + t_nh_n \in g^{-1}(\Theta)$ for all n. On the other hand, the strict differentiability assumption implies that

$$g(x_n + t_nh_n) = y_n + g'(\bar{x})(t_nh_n) + t_n\|h_n\|\alpha_n, \tag{3.8}$$

where $\{\alpha_n\}$ is a sequence in Y converging to 0. Since $g'(\bar{x})$ is surjective, the open mapping theorem implies that there exists a sequence $\{u_n\}$ in X converging to 0 such that $g'(\bar{x})(u_n) = \alpha_n$. Hence $g'(\bar{x})(h_n + \|h_n\|u_n) \to g'(\bar{x})(h)$. Noting (by (3.8)) that

$$y_n + t_ng'(\bar{x})(h_n + \|h_n\|u_n) = g(x_n + t_nh_n) \in \Theta,$$

it follows that $g'(\bar{x})(h) \in T_c(\Theta, g(\bar{x}))$. Therefore,

$$T_c(g^{-1}(\Theta),\bar{x}) \subset (g'(\bar{x}))^{-1}(T_c(\Theta,g(\bar{x}))).$$

Conversely, let $u \in (g'(\bar{x}))^{-1}(T_c(\Theta,g(\bar{x})))$. Then $g'(\bar{x})(u) \in T_c(\Theta,g(\bar{x}))$. To prove (3.6), we have to show that $u \in T_c(g^{-1}(\Theta),\bar{x})$. To do this, let $x_n \xrightarrow{g^{-1}(\Theta)} \bar{x}$ and $t_n \searrow 0$. Then $g(x_n) \xrightarrow{\Theta} g(\bar{x})$. Hence, there exists a sequence $v_n \to g'(\bar{x})(u)$ such that $g(x_n) + t_nv_n \in \Theta$ for all n. By the Lyusternik–Graves theorem, we assume without loss of generality that

$$d(x_n + t_nu, g^{-1}(g(x_n) + t_nv_n)) \le \mu\|g(x_n + t_nu) - g(x_n) - t_nv_n\| \tag{3.9}$$

for some $\mu \in (0,\ +\infty)$ and all $n \in \mathbb{N}$. By the strict differentiability of g at $\bar{x}$,

$$g(x_n + t_nu) - g(x_n) = g'(\bar{x})(t_nu) + o(t_n).$$

This and (3.9) imply that there exists $\tilde{x}_n$ with

$$\tilde{x}_n \in g^{-1}(g(x_n) + t_nv_n) \subset g^{-1}(\Theta)$$

such that

$$\|x_n + t_nu - \tilde{x}_n\| \le 2\mu(t_n\|g'(\bar{x})(u) - v_n\| + \|o(t_n)\|).$$

Then $u_n := \frac{\tilde{x}_n - x_n}{t_n} \to u$ and $x_n + t_nu_n = \tilde{x}_n \in g^{-1}(\Theta)$. This shows that $u \in T_c(g^{-1}(\Theta),\bar{x})$ as required to show. Since $g'(\bar{x})(X) = Y$, (3.7) is immediate from (3.6) and [39, Corollary 2.8.4 (ii)] (applied to $g'(\bar{x}), X, T_c(\Theta, g(\bar{x}))$ in place of A, L, M). The proof is complete.

Remark 3.1. Our formula (3.7) was inspired by Mordukhovich [24, Corollary 1.15] where the same relation was established but for Fréchet normal cones in place of Clarke normal cones. In the literature, study on the calculus of the Clarke tangent cone and normal cone seems to be quite scarce. Nevertheless, Clarke [4, p. 108, Corollary 1] did prove the same formula but required that $g'(\bar{x})(X) \cap \mathrm{int}(T_c(\Theta, g(\bar{x})) \neq \emptyset$ and that Θ admit a hypertangent vector at $g(\bar{x})$, namely, there exist $v \in Y$ and $r > 0$ such that

$$B(g(\bar{x}), r) \cap \Theta + tB(v, r) \subset \Theta \ \ \forall t \in (0,\ r).$$

For Proposition 3.4, we shall also need the following lemma.

LEMMA 3.6. *Let $g : X \to Z$ be smooth and $a \in X$, and suppose that $g'(a)$ is surjective. Then there exist $l, r \in (0,\ +\infty)$ such that*

$$lB_Z \subset g'(u)(B_X) \ \ \text{and} \ \ l\|z^*\| \leq \|(g'(u))^*(z^*)\| \quad \forall u \in B(a, r) \text{ and } \forall z^* \in Z^*. \tag{3.10}$$

Proof. We need only show that the inclusion in (3.10) holds for some $l, r \in (0,\ +\infty)$ (the inequality then follows easily). By the surjectivity assumption and the open mapping theorem, there exists $l \in (0,\ +\infty)$ such that $2lB_Z \subset g'(a)(B_X)$; by the smoothness of g, there exists $r > 0$ such that $\|g'(u) - g'(a)\| < \frac{l}{2}$ for all $u \in B(a, r)$. Hence,

$$2lB_Z \subset (g'(u) + (g'(a) - g'(u)))(B_X) \subset g'(u)(B_X) + \frac{l}{2}B_Z \quad \forall u \in B(a, r).$$

By the Radstrom cancellation lemma (cf. [42, Lemma 2.3]), this implies that

$$\frac{3l}{2}B_Z \subset \mathrm{cl}(g'(u)(B_X)) \quad \forall u \in B(a, r). \tag{3.11}$$

Since X, Z are Banach spaces and $g'(u)$ is a bounded linear operator from X to Z, $g'(u)(B_X)$ and $\mathrm{cl}(g'(u)(B_X))$ have the same interior (by [15, p. 183, Theorem A.1]). It follows from (3.11) that the inclusion in (3.10) holds. This completes the proof.

Proof of Proposition 3.4. We shall prove only the assertion regarding the $\mathcal{L}$-subsmoothness (the corresponding assertion regarding the subsmoothness can be proved similarly). By the smoothness and surjectivity assumption and Lemma 3.6, there exist $M, l, r \in (0,\ +\infty)$ such that (3.2) and (3.10) hold. Suppose that G is $\mathcal{L}$-subsmooth at $(g(a), b)$. Let $\varepsilon > 0$ and $\sigma := \frac{l\varepsilon}{(l+1)(M+1)}$. Then there exists $\eta > 0$ such that

$$\langle w^*, z - w\rangle + \langle v^*, y - b\rangle \leq \sigma(\|z - w\| + \|y - b\|)$$

for any $w \in G^{-1}(b) \cap B(g(a), \eta)$, $(w^*, v^*) \in N_c(\mathrm{Gr}(G), (w, b)) \cap (B_{Z^*} \times B_{Y^*})$, and $(z, y) \in \mathrm{Gr}(G)$ with $\|z - g(a)\| + \|y - b\| \leq \eta$. On the other hand, the smoothness of g implies that there exists $\delta \in (0,\ r)$ such that

$$\|g(x) - g(u) - g'(u)(x - u)\| \leq \sigma\|x - u\| \quad \forall x, u \in B(a, \delta), \tag{3.12}$$

$$g(B(a, \delta)) \subset B(g(a), \eta),$$

and

$$\|x - a\| + \|y - b\| < \delta \Longrightarrow \|g(x) - g(a)\| + \|y - b\| < \eta.$$

Let $u \in F^{-1}(b) \cap B(a, \delta)$. Then $g(u) \in G^{-1}(b) \cap B(g(a), \eta)$ and hence

$$\langle w^*, g(x) - g(u)\rangle + \langle v^*, y - b\rangle \leq \sigma(\|g(x) - g(u)\| + \|y - b\|) \tag{3.13}$$

for any $(w^*, v^*) \in N_c(\mathrm{Gr}(G), (g(u), b)) \cap (B_{Z^*} \times B_{Y^*})$ and $(x, y) \in \mathrm{Gr}(F)$ with $\|x - a\| + \|y - b\| < \delta$. Let $\tilde{g} : X \times Y \to Z \times Y$ be defined by $\tilde{g}(x, y) = (g(x), y)$ for all $(x, y) \in X \times Y$. Then $\tilde{g}$ is smooth and $\tilde{g}'(u, b)(B_X \times B_Y) = g'(u)(B_X) \times B_Y$; hence $\tilde{g}'(u, b)$ is surjective (by the first equality of (3.10)). Noting that $\mathrm{Gr}(F) = \tilde{g}^{-1}(\mathrm{Gr}(G))$, it follows from Lemma 3.5 that $N_c(\mathrm{Gr}(F), (u, b)) = (\tilde{g}'(u, b))^*(N_c(\mathrm{Gr}(G), (g(u), b)))$. This and the definition of $\tilde{g}$ imply that

$$N_c(\mathrm{Gr}(F), (u, b)) = \{((g'(u))^*(z^*), y^*) : \ (z^*, y^*) \in N_c(\mathrm{Gr}(G), (g(u), b))\}.$$

Now let $(u^*, y^*) \in N_c(\mathrm{Gr}(F), (u, b)) \cap (B_{X^*} \times B_{Y^*})$. Then there exists $z^* \in Z^*$ such that $u^* = (g'(u))^*(z^*)$ and $(z^*, y^*) \in N_c(\mathrm{Gr}(G), (g(u), b))$. It follows from (3.10) that $\|z^*\| \leq \frac{1}{l}$. Thus, applying (3.13) (with $\frac{l}{1+l}(z^*, y^*)$ in place of (w^*, v^*)) and making use of (3.2), one has

$$\langle z^*, g(x) - g(u)\rangle + \langle y^*, y - b\rangle \leq \frac{\sigma(l+1)}{l}(\|g(x) - g(u)\| + \|y - b\|) \leq \frac{\sigma(l+1)}{l}(M\|x - u\| + \|y - b\|) \tag{3.14}$$

for any $(x, y) \in \mathrm{Gr}(F)$ with $\|x - a\| + \|y - b\| < \delta$. Moreover, (3.12) entails that for any $x \in B(a, \delta)$,

$$-\frac{\sigma}{l}\|x - u\| \leq \langle z^*, g(x) - g(u) - g'(u)(x - u)\rangle = \langle z^*, g(x) - g(u)\rangle - \langle u^*, x - u\rangle.$$

This and (3.14) imply that

$$\langle u^*, x - u\rangle + \langle y^*, y - b\rangle \leq \frac{\sigma(l+1)(M+1)}{l}(\|x - u\| + \|y - b\|) = \varepsilon(\|x - u\| + \|y - b\|)$$

for any $(x, y) \in \mathrm{Gr}(F)$ with $\|x - a\| + \|y - b\| < \delta$. This shows that F is $\mathcal{L}$-subsmooth at (a, b). The proof is complete.

Note that every closed convex multifunction is subsmooth at each point of its graph. The following corollary is immediate from Proposition 3.4.

COROLLARY 3.7. *Suppose that F is defined by $F = G \circ g$, namely, $F(x) = G(g(x))$ for all $x \in X$, where $g : X \to Z$ is a smooth function and $G : Z \rightrightarrows Y$ is a closed convex multifunction. Let $(a, b) \in \mathrm{Gr}(F)$ and suppose that $g'(a)$ is surjective. Then F is subsmooth at (a, b).*

4. Calmness for multifunctions. Throughout this section, let $M : Y \rightrightarrows X$ be a closed multifunction. We also fix (arbitrary) $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$.

It is easy to verify that M is calm at $(\bar{y}, \bar{x})$ if and only if there exist $\tau, \delta \in (0, +\infty)$ such that

$$M(y) \cap B(\bar{x}, \delta) \subset M(\bar{y}) + \tau\|y - \bar{y}\|B_X \ \ \forall\ y \text{ close to } \bar{y}. \tag{4.1}$$

Motivated by the notion of linear cover property (cf. [7, 23, 28]), let us say that a multifunction $\Phi : X \rightrightarrows Y$ has the linear cover-like property at $(\bar{x}, \bar{y})$ if there exists $\tau \in (0, +\infty)$ such that for all x close to $\bar{x}$ and $r > 0$

$$\bar{y} \in \Phi(x) + \mathrm{int}(rB_Y) \implies \bar{y} \in \Phi(x + \mathrm{int}(\tau r B_X)). \tag{4.2}$$

In terms of (4.1) and (4.2), the following proposition provides formulas for the calmness modulus $\eta(M;\bar{y},\bar{x})$ (which is defined by (1.5)); we omit its proof as it is immediate from the related definitions.

PROPOSITION 4.1.

$$\begin{aligned}\eta(M;\bar{y},\bar{x}) &= \inf\{\tau>0:\ (4.1)\text{ holds for some }\delta>0\}\\ &= \inf\{\tau>0:\ (4.2)\text{ holds with }\Phi=M^{-1}\ \forall\ r>0\text{ and }\forall\ x\text{ close to }\bar{x}\}.\end{aligned}$$

The remainder of this section is devoted to a study on the duality aspect of the calmness. We divide our discussion into two subsections addressing the necessary conditions and the sufficient conditions for calmness.

4.1. Necessary conditions for calmness. There are two results in this subsection: one is on the Banach space setting and the other on the Asplund spaces.

THEOREM 4.2. *Suppose that there exist $\eta,\delta\in(0,\ +\infty)$ such that* (1.1) *holds. Then*

$$\hat{N}(M(\bar{y}),u)\cap B_{X^*}\subset\eta D_c^*M^{-1}(u,\bar{y})(B_{Y^*})\quad\forall u\in M(\bar{y})\cap B(\bar{x},\delta). \tag{4.3}$$

Proof. Let $\mathcal{I}_{\mathrm{Gr}(M^{-1})}$ denote the indicator function of $\mathrm{Gr}(M^{-1})$. Then (1.1) can be rewritten as

$$d(x,M(\bar{y}))\le\mathcal{I}_{\mathrm{Gr}(M^{-1})}(x,y)+\eta\|y-\bar{y}\|\quad\forall(x,y)\in B(\bar{x},\delta)\times B(\bar{y},\delta). \tag{4.4}$$

Let $u\in M(\bar{y})\cap B(\bar{x},\delta)$ and $u^*\in\hat{N}(M(\bar{y}),u)\cap B_{X^*}$. Noting (cf. [24, Corollary 1.96]) that $\hat{N}(M(\bar{y}),u)\cap B_{X^*}=\hat{\partial}d(\cdot,M(\bar{y})))(u)$, it follows that for any $\sigma>0$ there exists $r\in(0,\ \delta)$ such that $B(u,r)\subset B(\bar{x},\delta)$ and

$$\langle u^*,x-u\rangle\le d(x,M(\bar{y}))+\sigma\|x-u\|\quad\forall x\in B(u,r). \tag{4.5}$$

Hence, by (4.4),

$$\langle u^*,x-u\rangle\le\mathcal{I}_{\mathrm{Gr}(M^{-1})}(x,y)+\eta\|y-\bar{y}\|+\sigma\|x-u\|\quad\forall(x,y)\in B(u,r)\times B(\bar{y},\delta),$$

that is, $(u,\bar{y})$ is a local minimizer of ϕ defined by

$$\phi(x,y):=-\langle u^*,x-u\rangle+\mathcal{I}_{\mathrm{Gr}(M^{-1})}(x,y)+\eta\|y-\bar{y}\|+\sigma\|x-u\|\quad\forall(x,y)\in X\times Y.$$

Hence, $(0,0)\in\partial_c\phi(u,\bar{y})$. It follows from [4, Theorem 2.9.8] that

$$(0,0)\in(-u^*,0)+N_c(\mathrm{Gr}(M^{-1}),(u,\bar{y}))+\{0\}\times\eta B_{Y^*}+(\sigma B_{X^*})\times\{0\},$$

that is,

$$(u^*+\sigma x_\sigma^*,-\eta y_\sigma^*)\in N_c(\mathrm{Gr}(M^{-1}),(u,\bar{y}))$$

for some $x_\sigma^*\in B_{X^*}$ and $y_\sigma^*\in B_{Y^*}$. Since B_{Y^*} is weak* compact, without loss of generality we can assume $(u^*+\sigma x_\sigma^*,-\eta y_\sigma^*)\xrightarrow{w^*}(u^*,-\eta v^*)$ for some v^* in B_{Y^*} as $\sigma\to0^+$. Hence $(u^*,-\eta v^*)\in N_c(\mathrm{Gr}(M^{-1}),(u,\bar{y}))$ (because $N_c(\mathrm{Gr}(M^{-1}),(u,\bar{y}))$ is weak*-closed). This implies that

$$u^*\in D_c^*M^{-1}(u,\bar{y})(\eta v^*)\subset\eta D_c^*M^{-1}(u,\bar{y})(B_{Y^*}).$$

This shows that (4.3) holds. The proof is complete.

When Y, X are Asplund spaces, the conclusion in Theorem 4.2 can be strengthened with $\hat{N}(M(\bar{y}), u)$ and $D_c^* M^{-1}(u, \bar{y})$ replaced, respectively, by $N(M(\bar{y}), u)$ and $D^* M^{-1}(u, \bar{y})$.

THEOREM 4.3. *Suppose that Y, X are Asplund spaces and that there exist $\eta, \delta \in (0, +\infty)$ such that* (1.1) *holds. Then*

$$N(M(\bar{y}), u) \cap B_{X^*} \subset \eta D^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, \delta). \tag{4.6}$$

Proof. Let $u \in M(\bar{y}) \cap B(\bar{x}, \delta)$ and $u^* \in B_{X^*} \cap N(M(\bar{y}), u)$. Then there exist sequences $\{u_n\}$ in $M(\bar{y}) \cap B(\bar{x}, \delta)$ and $\{u_n^*\}$ in X^* such that

$$u_n \to u, \quad u_n^* \xrightarrow{w^*} u^*, \text{ and } u_n^* \in \hat{N}(M(\bar{y}), u_n) \quad \forall n \in \mathbb{N}.$$

Similar to the proof of (4.5), there exists $r \in (0, \delta)$ such that $B(u_n, r) \subset B(\bar{x}, \delta)$ and

$$\langle u_n^*, x - u_n \rangle \le d(x, M(\bar{y})) + \frac{1}{n}\|x - u_n\| \quad \forall x \in B(u_n, r).$$

Letting

$$\phi(x, y) := -\langle u_n^*, x - u_n \rangle + \mathcal{I}_{\mathrm{Gr}(M^{-1})}(x, y) + \eta\|y - \bar{y}\| + \frac{1}{n}\|x - u_n\| \quad \forall (x, y) \in X \times Y,$$

from the corresponding part of the proof of Theorem 4.2, it follows that $(u_n, \bar{y})$ is a local minimizer of ϕ. This and [24, Theorem 2.33] imply that there exists $(w_n, y_n) \in \mathrm{Gr}(M^{-1})$ such that $\|w_n - u_n\| + \|y_n - \bar{y}\| < \frac{1}{n}$ and

$$(0, 0) \in (-u_n^*, 0) + \hat{N}(\mathrm{Gr}(M^{-1}), (w_n, y_n)) + \{0\} \times \eta B_{Y^*} + \frac{2}{n}(B_{X^*} \times B_{Y^*}).$$

Therefore, $(w_n, y_n) \to (u, \bar{y})$ and there exist $x_n^* \in B_{X^*}$ and $y_n^*, v_n^* \in B_{Y^*}$ such that

$$\left(u_n^* + \frac{2}{n}x_n^*, -\eta y_n^* - \frac{2}{n}v_n^*\right) \in \hat{N}(\mathrm{Gr}(M^{-1}), (w_n, y_n)).$$

Since B_{Y^*} is sequentially weak*-compact (as Y is an Asplund space), we can assume that $y_n^* \xrightarrow{w^*} y^* \in B_{Y^*}$ as $n \to \infty$. It follows that $(u^*, -\eta y^*) \in N(\mathrm{Gr}(M^{-1}), (u, \bar{y}))$ and so $u^* \in D^* M^{-1}(u, \bar{y})(\eta y^*)$. Therefore, (4.6) holds. The proof is complete.

Remark 4.1. In Asplund spaces, the limiting subdifferential enjoys, like the Clarke subdifferential, the full sum rule, but, on the other hand, the Mordukhovich normal cone is not necessarily weak*-closed. This is why the last part of the proof of Theorem 4.3 differs from that of Theorem 4.2.

4.2. Sufficient conditions for calmness of L-subsmooth multifunctions. Under a suitable L-subsmoothness assumption, we show in the next result that a slightly stronger condition than (4.3) turns out to be sufficient for calmness.

THEOREM 4.4. *Suppose that M is L-subsmooth (resp., weakly L-subsmooth) at $(\bar{y}, \bar{x})$ and that there exist $\eta, \delta \in (0, +\infty)$ such that*

$$N_c(M(\bar{y}), u) \cap B_{X^*} \subset \eta D_c^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in \mathrm{bd}(M(\bar{y})) \cap B(\bar{x}, \delta) \tag{4.7}$$

$$(\textit{resp., } N_c(M(\bar{y}), u) \cap B_{X^*} \subset \eta D^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in \mathrm{bd}(M(\bar{y})) \cap B(\bar{x}, \delta)).$$

Then M is calm at $(\bar{y}, \bar{x})$ and, more precisely, for any $\varepsilon \in (0, \frac{1}{1+\eta})$ there exists $\delta_\varepsilon > 0$ such that

$$d(x, M(\bar{y})) \leq \frac{\eta + (1+\eta)\varepsilon}{1 - (1+\eta)\varepsilon}\|y - \bar{y}\| \quad \forall y \in B(\bar{y}, \delta_\varepsilon) \text{ and } \forall x \in M(y) \cap B(\bar{x}, \delta_\varepsilon). \tag{4.8}$$

Proof. We provide only the proof for the assertion under the L-subsmoothness assumption (the proof for the other part is similar). Let $\varepsilon \in (0, \frac{1}{1+\eta})$. Then, by the L-subsmoothness assumption, there exists $\delta_\varepsilon \in (0, \frac{\delta}{2})$ such that

$$-\langle v^*, y - \bar{y}\rangle + \langle u^*, x - u\rangle \leq \varepsilon(\|y - \bar{y}\| + \|x - u\|) \tag{4.9}$$

whenever $y \in B(\bar{y}, 2\delta_\varepsilon)$, $x \in M(y) \cap B(\bar{x}, 2\delta_\varepsilon)$, $u \in M(\bar{y}) \cap B(\bar{x}, 2\delta_\varepsilon)$, $v^* \in B_{Y^*}$, and $u^* \in D_c^* M^{-1}(u, \bar{y})(v^*) \cap B_{X^*}$. To verify (4.8), let $y \in B(\bar{y}, \delta_\varepsilon)$ and $x \in M(y) \cap (B(\bar{x}, \delta_\varepsilon) \setminus M(\bar{y}))$. Then $d(x, M(\bar{y})) \leq \|x - \bar{x}\| < \delta_\varepsilon$. Let

$$\gamma \in \left(\max\left\{\frac{d(x, M(\bar{y}))}{\delta_\varepsilon}, (1+\eta)\varepsilon, \frac{1}{2}\right\},\ 1\right).$$

By Lemma 2.1 there exist $u \in \mathrm{bd}(M(\bar{y}))$ and $u^* \in N_c(M(\bar{y}), u)$ with $\|u^*\| = 1$ such that

$$\gamma\|x - u\| \leq \min\{\langle u^*, x - u\rangle, d(x, M(\bar{y}))\}. \tag{4.10}$$

Thus, $\|x - u\| \leq \frac{d(x, M(\bar{y}))}{\gamma} < \delta_\varepsilon$. Hence

$$\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| < 2\delta_\varepsilon < \delta.$$

By (4.7), there exists $v^* \in \eta B_{Y^*}$ such that $u^* \in D_c^* M^{-1}(u, \bar{y})(v^*)$. Applying (4.9) with $(\frac{u^*}{1+\eta}, \frac{v^*}{1+\eta})$ in place of (u^*, v^*), it follows that

$$-\langle v^*, y - \bar{y}\rangle + \langle u^*, x - u\rangle \leq (1+\eta)\varepsilon(\|y - \bar{y}\| + \|x - u\|)$$

and so

$$\begin{aligned}\langle u^*, x - u\rangle - (1+\eta)\varepsilon\|x - u\| &\leq \langle v^*, y - \bar{y}\rangle + (1+\eta)\varepsilon\|y - \bar{y}\| \\ &\leq (\eta + (1+\eta)\varepsilon)\|y - \bar{y}\|.\end{aligned}$$

This and (4.10) imply that

$$(\gamma - (1+\eta)\varepsilon)\|x - u\| \leq (\eta + (1+\eta)\varepsilon)\|y - \bar{y}\|$$

and hence

$$d(x, M(\bar{y})) \leq \frac{\eta + (1+\eta)\varepsilon}{\gamma - (1+\eta)\varepsilon}\|y - \bar{y}\|$$

(because $u \in M(\bar{y})$). Letting $\gamma \to 1$, it follows that (4.8) holds. The proof is complete.

The following example shows that the L-subsmoothness assumption cannot be dropped in Theorem 4.4.

Example 4.5. Let $X = Y = R$ and let

$$\Omega_1 = \{(s,t) \in R^2 : \ s^2 + (t-1)^2 \le 1 \text{ and } (s-1)^2 + t^2 \le 1\},$$
$$\Omega_2 = \{(s,t) \in R^2 : \ s^2 + (t+1)^2 \le 1 \text{ and } (s-1)^2 + t^2 \le 1\},$$
$$\Omega_3 = \{(s,t) \in R^2 : \ (s+1)^2 + t^2 \le 1 \text{ and } s^2 + (t+1)^2 \le 1\},$$
$$\Omega_4 = \{(s,t) \in R^2 : \ (s+1)^2 + t^2 \le 1 \text{ and } s^2 + (t-1)^2 \le 1\}.$$

Define the multifunction $M : Y \rightrightarrows X$ such that $\mathrm{Gr}(M) = \bigcup_{i=1}^4 \Omega_i$. Then $M(0) = \{0\}$ and so $N_c(M(0), 0) = X^*$. It is easy to verify that $N_c(\mathrm{Gr}(M), (0,0)) = X^* \times Y^*$. Hence

$$N_c(M(0), 0) \cap B_{X^*} = B_{X^*} \subset \tau D_c^* M^{-1}(0,0)(B_{Y^*}) = X^* \quad \forall \tau \in (0, \ +\infty).$$

On the other hand, note that

$$\frac{1}{n} \in M\left(\frac{1}{n^2\left(1+\sqrt{1-\frac{1}{n^2}}\right)}\right) \quad \text{and} \quad \frac{d(\frac{1}{n}, M(0))}{\left\|\frac{1}{n^2\left(1+\sqrt{1-\frac{1}{n^2}}\right)} - 0\right\|} \to +\infty.$$

Hence M is not calm at $(0,0)$.

Recall [43] that M is strongly calm at $(\bar{y}, \bar{x})$ if there exist $\eta, \delta \in (0, \ +\infty)$ such that

$$\|x - \bar{x}\| \le \eta \|y - \bar{y}\| \quad \forall y \in B(\bar{y}, \delta) \text{ and } x \in M(y) \cap B(\bar{x}, \delta).$$

It is clear that M is strongly calm at $(\bar{y}, \bar{x})$ if and only if $\bar{x}$ is an isolated point of $M(\bar{y})$ (i.e., $M(\bar{y}) \cap B(\bar{x}, r) = \{\bar{x}\}$ for some $r > 0$) and M is calm at $(\bar{y}, \bar{x})$.

COROLLARY 4.6. *Suppose that M satisfies condition* (S) *at $(\bar{y}, \bar{x})$. Then M is strongly calm at $(\bar{y}, \bar{x})$ if and only if*

$$D_c^* M^{-1}(\bar{x}, \bar{y})(Y^*) = X^*. \tag{4.11}$$

Proof. Suppose that (4.11) holds. Since $D_c^* M^{-1}(\bar{x}, \bar{y})$ is a closed convex multifunction from Y^* to X^*, (4.11) and the Robinson–Ursescu theorem (cf. [33, 36]) imply that there exists $\eta > 1$ such that

$$\frac{1}{\eta} B_{X^*} \subset D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) \cap B_{X^*}. \tag{4.12}$$

Hence, by the Hahn–Banach theorem,

$$\frac{1}{\eta}\|u\| \le \max\{\langle u^*, u\rangle : \ u^* \in D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) \cap B_{X^*}\} \quad \forall u \in X. \tag{4.13}$$

Consider $\varepsilon \in (0, \ \frac{1}{\eta})$. The condition (S) assumption implies that there exists $\delta > 0$ such that

$$\langle u^*, x - \bar{x}\rangle \le \varepsilon \|x - \bar{x}\| \quad \forall x \in M(\bar{y}) \cap B(\bar{x}, \delta) \text{ and } \forall u^* \in D_c^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) \cap B_{X^*}.$$

It follows from (4.13) that $M(\bar{y}) \cap B(\bar{x}, \delta) = \{\bar{x}\}$. This entails that M is L-subsmooth at $(\bar{y}, \bar{x})$ and (4.7) holds (due to the condition (S) assumption and (4.12), respectively). Therefore, Theorem 4.4 can be applied to conclude that M is calm at $(\bar{y}, \bar{x})$.

Conversely, suppose that M is strongly calm at $(\bar{y}, \bar{x})$. Then $M(\bar{y}) \cap B(\bar{x}, r) = \{\bar{x}\}$ for some $r > 0$ (so $\hat{N}(M(\bar{y}), \bar{x}) = X^*$), and Theorem 4.2 implies that there exist $\eta, \delta \in (0, +\infty)$ such that (4.3) holds and so does (4.11). The proof is complete.

COROLLARY 4.7. *Suppose that Y, X are finite-dimensional and that* $\mathrm{Gr}(M)$ *is Clarke regular at $(\bar{y}, \bar{x})$ (i.e., $T_c(\mathrm{Gr}(M), (\bar{y}, \bar{x})) = T(\mathrm{Gr}(M), (\bar{y}, \bar{x}))$). Then M is strongly calm at $(\bar{y}, \bar{x})$ if and only if $D_c^* M^{-1}(\bar{x}, \bar{y})(Y^*) = X^*$.*

Proof. By Corollary 4.6, we need only show that M satisfies condition (S) at $(\bar{y}, \bar{x})$. To do this, suppose to the contrary that there exist $\varepsilon_0 > 0$, a sequence $\{(y_n, x_n)\}$ in $\mathrm{Gr}(M) \setminus \{(\bar{y}, \bar{x})\}$, and a sequence $\{(v_n^*, u_n^*)\}$ in $N_c(\mathrm{Gr}(M), (\bar{y}, \bar{x})) \cap (B_{Y^*} \times B_{X^*})$ such that $(y_n, x_n) \to (\bar{y}, \bar{x})$ and

$$\langle v_n^*, y_n - \bar{y} \rangle + \langle u_n^*, x_n - \bar{x} \rangle > \varepsilon_0(\|y_n - \bar{y}\| + \|x_n - \bar{x}\|) \quad \forall n.$$

Since Y, X are finite-dimensional, we can assume that

$$\frac{(y_n - \bar{y}, x_n - \bar{x})}{\|y_n - \bar{y}\| + \|x_n - \bar{x}\|} \to (v, u) \text{ and } (v_n^*, u_n^*) \to (v^*, u^*)$$

for some $(v, u) \in Y \times X$ and $(v^*, u^*) \in Y^* \times X^*$. Then $\langle v^*, v \rangle + \langle u^*, v \rangle \geq \varepsilon_0$, $(v, u) \in T(\mathrm{Gr}(M), (\bar{y}, \bar{x}))$, and $(v^*, u^*) \in N_c(\mathrm{Gr}(M), (\bar{y}, \bar{x}))$. This contradicts the Clarke regularity assumption. The proof is complete.

When X is an Asplund space, the assumption in Theorem 4.4 can be weakened with $N_c(M(\bar{y}), u)$ replaced by $\hat{N}(M(\bar{y}), u)$.

THEOREM 4.8. *Suppose that X is an Asplund space and that M is L-subsmooth (resp., weakly L-subsmooth) at $(\bar{y}, \bar{x})$ and that there exist $\eta, \delta \in (0, +\infty)$ such that*

$$\hat{N}(M(\bar{y}), u) \cap B_{X^*} \subset \eta D_c^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, \delta)$$

$$(\textit{resp.}, \hat{N}(M(\bar{y}), u) \cap B_{X^*} \subset \eta D^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, \delta)).$$

Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that (4.8) *holds.*

The proof of Theorem 4.8 is the same as that of Theorem 4.4, but with the Asplund space version of Lemma 2.1 applied in place of the Banach space version.

Making use of (2.2) and the equivalence

$$x^* \in (D_c^* M(\bar{y}, u))^{-1}(y^*) \Leftrightarrow -x^* \in D_c^* M^{-1}(u, \bar{y})(-y^*),$$

it is easy from Theorem 4.2 to verify part (i) of the following corollary. Similarly, part (ii) follows from Theorem 4.4.

COROLLARY 4.9. *The following assertions hold.*

(i) $\eta(M; \bar{y}, \bar{x}) \geq \limsup_{u \in M(\bar{y}), u \to \bar{x}} \|D_c^* M(\bar{y}, u)|_{-\hat{N}(M(\bar{y}), u)}\|^-$.

(ii) *If M is L-subsmooth at $(\bar{y}, \bar{x})$, then*

$$\eta(M; \bar{y}, \bar{x}) \leq \limsup_{u \in M(\bar{y}), u \to \bar{x}} \|D_c^* M(\bar{y}, u)|_{-N_c(M(\bar{y}), u)}\|^-.$$

Similarly, one can use Theorems 4.3 and 4.8 to show the following corollary.

COROLLARY 4.10. *Suppose that Y, X are Asplund spaces. Then the following assertions hold.*

(i) $\eta(M; \bar{y}, \bar{x}) \geq \limsup_{u \in M(\bar{y}), u \to \bar{x}} \|D^* M(\bar{y}, u)|_{-\hat{N}(M(\bar{y}), u)}\|^-$.

(ii) *If M is weakly L-subsmooth at $(\bar{y}, \bar{x})$, then the equality in* (i) *holds.*

Note that, in the Asplund space setting, the Fréchet normal cone $\hat{N}(M(\bar{y}), u)$ is used in Theorem 4.8. However, for the general Banach spaces, one needs to use the

Clarke normal cone $N_c(M(\bar{y}), u)$ in Theorem 4.4; we have only the inequality version in Corollary 4.9(ii) while we have the equality version in Corollary 4.10(ii).

The following result concerns "convex-composite" multifunctions. Optimization problems involving convex-composite functions have been extensively studied (for details, see [16, 30, 35, 37] and the references therein).

THEOREM 4.11. *Suppose that M is defined by $M = g^{-1} \circ G$, namely, $M(y) = g^{-1}(G(y))$ for all $y \in Y$, where $G : Y \rightrightarrows Z$ is a closed convex multifunction and $g : X \to Z$ is a smooth function. Let $(\bar{y}, \bar{x}) \in \mathrm{Gr}(M)$ and suppose that $g'(\bar{x})$ is surjective. Then the following assertions hold.*

(i) $\eta(M; \bar{y}, \bar{x}) = \limsup_{u \in M(\bar{y}), u \to \bar{x}} \|D^*M(\bar{y}, u)|_{-N(M(\bar{y}),u)}\|^-$.

(ii) *If there exist $\eta, \delta \in (0, +\infty)$ such that*

$$N(M(\bar{y}), u) \cap B_{X^*} \subset \eta D^* M^{-1}(u, \bar{y})(B_{Y^*}) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, \delta), \tag{4.14}$$

then for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \delta)$ such that

$$d(x, M(\bar{y})) \leq \frac{\eta}{1 - \varepsilon} \|y - \bar{y}\| \tag{4.15}$$

whenever $y \in B(\bar{y}, \delta_\varepsilon)$ and $x \in M(y) \cap B(\bar{x}, \delta_\varepsilon)$.

Proof. By Corollary 3.7, M^{-1} $(= G^{-1} \circ g)$ is subsmooth at $(\bar{x}, \bar{y})$ and so is M at $(\bar{y}, \bar{x})$. Let $\tilde{g}(y, x) := (y, g(x))$ for all $(y, x) \in Y \times X$. Then it follows from the surjectivity of $g'(\bar{x})$ that $\tilde{g}'(\bar{y}, \bar{x})$ is surjective. By Lemma 3.6, take $r > 0$ such that $g'(u)$ and $\tilde{g}'(y, u)$ are surjective for all $(y, u) \in B(\bar{y}, r) \times B(\bar{x}, r)$. Noting that $M(\bar{y}) = g^{-1}(G(\bar{y}))$ and $\mathrm{Gr}(M) = \tilde{g}^{-1}(\mathrm{Gr}(G))$, it follows from Lemma 3.5 and [24, Theorem 1.17] that

$$\hat{N}(M(\bar{y}), u) = N(M(\bar{y}), u) = N_c(M(\bar{y}), u) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, r) \tag{4.16}$$

and

$$\hat{N}(\mathrm{Gr}(M), (y, u)) = N(\mathrm{Gr}(M), (y, u)) = N_c(\mathrm{Gr}(M), (y, u))$$

for all $(y, u) \in \mathrm{Gr}(M) \cap (B(\bar{y}, r) \times B(\bar{x}, r))$; hence

$$D^*M(\bar{y}, u) = D_c^*M(\bar{y}, u) \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, r).$$

Thus (i) follows from Corollary 4.9.

To prove (ii), let $\eta, \delta \in (0, +\infty)$ satisfy (4.14). Let $\varepsilon \in (0, 1)$. Then, by the subsmoothness of M at $(\bar{y}, \bar{x})$, there exists $\delta_\varepsilon \in (0, r)$ such that

$$\langle u^*, x - u \rangle - \langle v^*, y - \bar{y} \rangle \leq \varepsilon \|x - u\| \tag{4.17}$$

whenever $y \in B(\bar{y}, 2\delta_\varepsilon)$, $x \in M(y) \cap B(\bar{x}, 2\delta_\varepsilon)$, $u \in M(\bar{y}) \cap B(\bar{x}, 2\delta_\varepsilon)$, $v^* \in Y^*$, and $u^* \in D^*M^{-1}(u, \bar{y})(v^*) \cap B_{X^*}$. Let $y \in B(\bar{y}, \delta_\varepsilon)$ and $x \in B(\bar{x}, \delta_\varepsilon) \setminus M(\bar{y})$. We have to show that (4.15) holds. To do this, let $\gamma \in \left(\max\left\{\frac{d(x, M(\bar{y}))}{\delta_\varepsilon}, \ \frac{1}{2}\right\}, \ 1\right)$ sufficiently close to 1. By Lemma 2.1, as in the corresponding part of the proof of Theorem 4.4, there exist $u \in \mathrm{bd}(M(\bar{y})) \cap B(\bar{x}, 2\delta_\varepsilon)$ and $u^* \in N_c(M(\bar{y}), u)$ with $\|u^*\| = 1$ such that (4.10) holds. It follows from (4.14) and (4.16) that there exists $v^* \in \eta B_{Y^*}$ such that $u^* \in D^*M^{-1}(u, \bar{y})(v^*)$. By (4.10) and (4.17), one has

$$(\gamma - \varepsilon)\|x - u\| \leq \langle u^*, x - u \rangle - \varepsilon \|x - u\| \leq \langle v^*, y - \bar{y} \rangle \leq \eta \|y - \bar{y}\|$$

and so $(\gamma-\varepsilon)d(x, M(\bar{y})) \le \eta\|y-\bar{y}\|$. Letting $\gamma \to 1$, it follows that (4.15) holds. The proof is complete.

Remark 4.2. Motivated by (1.4), a natural problem is whether the upper limit

$$\limsup_{u\in M(\bar{y}),u\to\bar{x}} \|D_c^*M(\bar{y},u)|_{-N_c(M(\bar{y}),u)}\|^-$$

in Corollaries 4.9 and 4.10 and Theorem 4.11 can be replaced with

$$\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^-.$$

The answer is negative even when M is convex and X, Y are finite-dimensional. Below we give an example of a closed convex multifunction M between two finite-dimensional spaces such that

$$\|D_c^*M(\bar{x},\bar{y})|_{-N_c(M(\bar{y}),\bar{x})}\|^- = 0 \quad \text{but} \quad \eta(M;\bar{y},\bar{x}) = +\infty.$$

Let $S = \{(u,v) \in R^2 :\ u^2+v^2 \le 1\}$ and $C = \{(u,v) \in R^2 :\ v \le 1\}$. Let $M : R \rightrightarrows R^2$ be defined by

$$M(y) := \{x \in C :\ d^2(x,S) \le y\} \quad \forall y \in R.$$

Then M is a closed convex multifunction (because C and S are closed convex sets). Take $\bar{y} = 0$ and $\bar{x} = (0,1)$. Then $S = M(\bar{y})$ and $\bar{x} \in M(\bar{y})$. It is clear that

$$d(x, M(\bar{y})) = d(x,S) \ \text{ and } \ x \in M(d^2(x,S)) \quad \forall x \in C.$$

This shows that M is not calm at $(\bar{y},\bar{x})$, that is, $\eta(M;\bar{y},\bar{x}) = +\infty$. Next we show that $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x}))}\|^- = 0$. It is easy from the convexity of $M(\bar{y})$ to verify that

$$N_c(M(\bar{y}),\bar{x}) = N(S,\bar{x}) = N(C,\bar{x}) = \{0\} \times R_+.$$

Let $t, r \in (0,\ +\infty)$. Then, for any $x \in C$ and $y \in M^{-1}(x) = [d^2(x,S),\ +\infty)$, one has

$$\langle (0,t), x\rangle - r(y-\bar{y}) \le \langle (0,t), \bar{x}\rangle.$$

This and the convexity of M imply that $((0,t), -r) \in N_c(\mathrm{Gr}(M^{-1}), (\bar{x},\bar{y}))$ and so $(0,t) \in D_c^*M^{-1}(\bar{x},\bar{y})(r)$. Since t and r are arbitrary in $(0,\ +\infty)$, $N_c(M(\bar{y}),\bar{x}) \subset D_c^*M^{-1}(\bar{x},\bar{y})(r)$. This shows that

$$-N_c(M(\bar{y}),\bar{x}) \subset (D_c^*M(\bar{y},\bar{x}))^{-1}(-r) \subset (D_c^*M(\bar{y},\bar{x}))^{-1}(rB_{Y^*}) \ \ \forall r > 0.$$

It follows from (2.2) that $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- = 0$.

Nevertheless, in the convex-composite case, $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- < +\infty$ does imply the calmness of the sublinear multifunction $D_cM(\bar{y},\bar{x})$. First, we provide a result in a general case.

PROPOSITION 4.12. *Suppose that*

$$T_c(M(\bar{y}),\bar{x}) \subset D_cM(\bar{y},\bar{x})(0). \tag{4.18}$$

Then

$$\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- = \inf\{\eta > 0 :\ d(x, T_c(M(\bar{y}),\bar{x})) \le \eta\|y\| \ \ \forall y \in Y \ \textit{and}\ x \in D_cM(\bar{y},\bar{x})(y))\}.$$

If, in addition, $\|D_c^*M(\bar{y},\bar{x})_{-N_c(M(\bar{y}),\bar{x})}\|^- < +\infty$*, then*

$$T_c(M(\bar{y}),\bar{x}) = D_cM(\bar{y},\bar{x})(0)$$

and so $\eta(D_cM(\bar{y},\bar{x});0,0) = \|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y},\bar{x})}\|^-$.

Proof. Let

$$\eta_* := \inf\{\eta > 0 : \ d(x, T_c(M(\bar{y}),\bar{x})) \le \eta\|y\| \ \ \forall y \in Y \text{ and } x \in D_cM(\bar{y},\bar{x})(y))\}.$$

First, we assume that $\eta_* < \infty$. Consider any $\eta \in (\eta_*, \ \infty)$. Then

$$d(x, T_c(M(\bar{y},\bar{x})) \le \eta\|y\| \ \ \forall y \in Y \text{ and } x \in D_cM(\bar{y},\bar{x})(y).$$

It follows that $D_cM(\bar{y},\bar{x})(0) \subset T_c(M(\bar{y}),\bar{x})$. This and (4.18) imply that

$$T_c(M(\bar{y}),\bar{x}) = D_cM(\bar{y},\bar{x})(0).$$

Hence $\eta(D_c(M(\bar{y};\bar{x}),0,0) = \eta_*$. Noting that $D_cM(\bar{y},\bar{x})$ is a closed convex multifunction, it follows from [43, Theorem 4.3] that

$$\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- = \|D_c^*(D_cM(\bar{y},\bar{x}))(0,0)|_{-N_c(D_cM(\bar{y},\bar{x})(0),0)}\|^- = \eta_*.$$

It remains to show that $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- = +\infty$ if $\eta_* = +\infty$. Suppose that $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- < +\infty$. We need only show that $\eta_* < +\infty$. Let $x \in X \setminus T_c(M(\bar{y}),\bar{x})$ and $\gamma \in (0, \ 1)$. By Lemma 2.1 there exist

$$z \in T_c(M(\bar{y}),\bar{x}) \ \text{ and } \ z^* \in N_c(T_c(M(\bar{y}),\bar{x}),z)$$

such that

$$\|z^*\| = 1 \ \text{ and } \ \langle z^*, x - z\rangle \ge \gamma\|x - z\|. \tag{4.19}$$

Noting that $T_c(M(\bar{y}),\bar{x})$ is a convex cone, it is easy to verify that

$$N_c(T_c(M(\bar{y}),\bar{x}),z) \subset N_c(T_c(M(\bar{y}),\bar{x}),0) = N_c(M(\bar{y}),\bar{x}).$$

Therefore, $z^* \in N_c(M(\bar{y}),\bar{x})$. Let $\eta \in (\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^-, \ \infty)$. Then there exists $y^* \in D_c^*M(\bar{y},\bar{x})(-z^*)$ such that $\|y^*\| < \eta$. It follows that

$$(y^*, z^*) \in N_c(\mathrm{Gr}(M),(\bar{y},\bar{x})) = N_c(\mathrm{Gr}(D_cM(\bar{y},\bar{x})),(0,0)).$$

Since $\mathrm{Gr}(D_cM(\bar{y},\bar{x}))$ is convex,

$$\langle y^*, y - 0\rangle + \langle z^*, x - 0\rangle \le 0 \quad \forall (y,x) \in \mathrm{Gr}(D_cM(\bar{y},\bar{x})).$$

Noting that $\langle z^*, z\rangle = 0$ (because $z^* \in N_c(T_c(M(\bar{y}),\bar{x}),z)$ and $T_c(M(\bar{y}),\bar{x})$ is a convex cone), it follows from (4.19) that

$$\gamma d(x, T_c(M(\bar{y}),\bar{x})) \le \gamma\|x - z\| \le -\langle y^*, y\rangle \le \eta\|y\| \quad \forall (y,x) \in \mathrm{Gr}(D_cM(\bar{y},\bar{x})).$$

Letting $\gamma \to 1$, one has

$$d(x, T_c(M(\bar{y}),\bar{x})) \le \eta\|y\| \quad \forall (y,x) \in \mathrm{Gr}(D_cM(\bar{y},\bar{x})).$$

Hence, $\eta_* \le \eta < +\infty$. The proof is complete.

Remark 4.3. If one drops the condition $\|D_c^*M(\bar{y},\bar{x})|_{-N_c(M(\bar{y}),\bar{x})}\|^- < \infty$, it is possible that $T_c(M(\bar{y}),\bar{x}) \neq D_cM(\bar{y},\bar{x})(0)$. For example, let $M : R \rightrightarrows R$ be defined by

$$\mathrm{Gr}(M) := \{(y,x) \in R \times R : \ y^2 + x^2 \leq 1\}.$$

Thus, $M(1) = \{0\}$ and so $T_c(M(1),0) = \{0\}$; but $T_c(\mathrm{Gr}(M),(1,0)) = R_+ \times R$. Hence $D_cM(1,0)(0) = R$. This shows that $T_c(M(1),0) \neq D_cF(1,0)(0)$.

In Proposition 4.12, the assumption (4.18) is a mild one. Indeed, the corresponding assertion for contingent derivative always holds: $T(M(\bar{y}),\bar{x}) \subset DM(\bar{y},\bar{x})(0)$ (which is easy to verify). Thus, (4.18) is satisfied if $\mathrm{Gr}(M)$ is regular at $(\bar{y},\bar{x})$ in the Clarke sense. Hence, (4.18) is satisfied if M (resp., M^{-1}) is subsmooth at $(\bar{y},\bar{x})$ (resp., $(\bar{x},\bar{y})$); in particular, (4.18) is satisfied under the assumption of Theorem 4.11. Hence, the following corollary is immediate from Corollary 3.7 and Proposition 4.12.

Corollary 4.13. *Let G, g, M, and $(\bar{y},\bar{x})$ be as in Theorem* 4.11. *Then*

$$\|D^*M(\bar{y},\bar{x})|_{-N(M(\bar{y}),\bar{x})}\|^- = \inf\{\eta > 0 : \ d(x, T(M(\bar{y}),\bar{x})) \leq \eta\|y\| \ \ \forall y \in Y \text{ and } x \in DM(\bar{y},\bar{x})(y))\}.$$

If, in addition, $\|D^*M(\bar{y},\bar{x})_{-N(M(\bar{y}),\bar{x})}\|^- < +\infty$, then

$$\eta(DM(\bar{y},\bar{x}); 0, 0) = \|D^*M(\bar{y},\bar{x})|_{-N(M(\bar{y},\bar{x})}\|^-.$$

In what follows, we consider the multifunction $M : Y \rightrightarrows X$ defined by

$$M(y) := \{x \in X : \ g(x) + y \in \Lambda\} \quad \forall y \in Y, \tag{4.20}$$

where $g : X \to Y$ is a function and Λ is a closed subset of Y.

Theorem 4.14. *Let M be given by* (4.20) *and $(\bar{y},\bar{x}) \in \mathrm{Gr}(M)$. Suppose that g is smooth and that Λ is subsmooth at $g(\bar{x}) + \bar{y}$. Further suppose that there exist $\eta, \delta \in (0, +\infty)$ such that*

$$N_c(M(\bar{y}),u) \cap B_{X^*} \subset \eta(g'(u))^*(N_c(\Lambda, g(u) + \bar{y}) \cap B_{Y^*}) \quad \forall u \in M(\bar{y}) \cap B(\bar{x},\delta).$$

Then M is calm at $(\bar{y},\bar{x})$, and, more precisely, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d(x, M(\bar{y})) \leq \frac{\eta + (1+\eta)\varepsilon}{1 - (1+\eta)\varepsilon}\|y - \bar{y}\| \quad \forall y \in B(\bar{y},\delta_\varepsilon) \text{ and } \forall x \in M(y) \cap B(\bar{x},\delta_\varepsilon).$$

Proof. Note that $M^{-1}(x) = -g(x) + \Lambda$ for each $x \in X$. It follows from Proposition 3.3(ii) (applied to $M^{-1}, -g$ in place of F, g) that

$$D_c^*M^{-1}(u,\bar{y})(B_{Y^*}) = (g'(u))^*(N_c(\Lambda, g(u) + \bar{y}) \cap B_{Y^*}) \quad \forall u \in M(\bar{y}).$$

Similarly, since Λ is subsmooth at $g(\bar{x}) + \bar{y}$, Proposition 3.3(iii) implies that M^{-1} is subsmooth at $(\bar{x},\bar{y})$ and so is M at $(\bar{y},\bar{x})$. Thus, the assertion of Theorem 4.14 follows from Theorem 4.4. The proof is complete.

Let T be a compact topological space and let $\mathcal{C}(T)$ denote the Banach space of all continuous functions on T equipped with the sup-norm. Let $\psi : X \times T \to R$ be a function and consider the multifunction $M : \mathcal{C}(T) \rightrightarrows X$ defined by

$$M(y) := \{x \in X : \ \psi(x,t) \leq -y(t) \ \ \forall t \in T\} \quad \forall y \in \mathcal{C}(T); \tag{4.21}$$

equivalently one can write (4.21) as

$$M(y) = \{x \in X : \; g(x) + y \in \Lambda\} \quad \forall y \in \mathcal{C}(T),$$

where $g(x) = \psi(x, \cdot)$ and Λ is the convex cone of all nonpositive continuous functions on T. In the special case when $X = R^n$, $T \subset R^m$, and ψ is a continuously differentiable function on $R^n \times R^m$ such that $\psi_1'(x,t)$ is locally Lipschitzian on $R^n \times R^m$, where $\psi_1'(x,t)$ denotes the derivative of $\psi(x,t)$ with respect the first variable x, Henrion and Outrata [10] recently considered the calmness of M defined by (4.21) at $(0,\bar{x}) \in \mathrm{Gr}(M)$. For $x \in M(0)$, let $T(x) := \{t \in T : \; \psi(x,s) \le \psi(x,t) \text{ for all } s \in T\}$ and let

$$\mathcal{J} := \{S \in \mathcal{K}(T) : \; \exists x_i \overset{\mathrm{bd}M(0)\setminus\{\bar{x}\}}{\longrightarrow} \bar{x} \text{ s.t. } d_H(S, T(x_i)) \to 0\},$$

where $\mathcal{K}(T)$ denotes the family of all compact subsets of T and d_H denotes the Hausdorff distance between compact sets. Henrion and Outrata established the following sufficient condition for calmness (see [10, Theorem 4]).

THEOREM A. *Consider* (4.21) *with* $X = R^n$, $T \subset R^m$, *and* ψ *being a smooth function on* $R^n \times R^m$ *such that* $\psi_1'(x,t)$ *is locally Lipschitzian on* $R^n \times R^m$. *Let* $\bar{x} \in M(0)$ *with* $\psi(\bar{x},\bar{t}) = 0$ *for some* $\bar{t} \in T$. *Suppose that the following two conditions are satisfied.*

(1) $T(M(0),\bar{x}) = \{h \in R^n : \; \langle \psi_1'(\bar{x},t), h\rangle \le 0$ *for all* $t \in T(\bar{x})\}$.

(2) *There exists* $\rho > 0$ *such that* $d(0, \mathrm{co}\{\psi_1'(\bar{x},t) : \; t \in S\}) \ge \rho$ *for all* $S \in \mathcal{J}$.

Then M *is calm at* $(0,\bar{x})$.

Recently, Zheng and Yang [45] proved that the conditions (1) and (2) in Theorem A can be replaced by the following weaker condition: there exist $\eta, \delta \in (0, +\infty)$ such that

$$\text{(WC)} \quad N_c(M(0),u) \cap B_{X^*} \subset [0, \eta]\mathrm{co}\{\psi_1'(u,t) : \; t \in T(u)\} \quad \forall u \in M(0) \cap B(\bar{x},\delta).$$

As an application of Theorem 4.14, we can improve and generalize Theorem A to the general Banach space case. To do this, it would be convenient to recall some standard notation. Let $\mathcal{B}(T)$ denote the family of all Borel sets in T and let $\mathrm{rca}(T)$ denote the space of all regular finite real-valued Borel measures on T equipped with the total variation norm $\|\mu\| = |\mu|(T)$ for any $\mu \in \mathrm{rca}(T)$. Recall that a Borel measure μ on T is said to be supported on $A \in \mathcal{B}(T)$ if $\mu(B) = 0$ for all $B \in \mathcal{B}(T)$ with $B \cap A = \emptyset$. Let

$$\mathrm{rca}^+(T) := \{\mu \in \mathrm{rca}(T) : \; \mu(B) \ge 0 \;\; \forall B \in \mathcal{B}(T)\}$$

and

$$\mathrm{rac}_A^+(T) := \{\mu \in \mathrm{rac}^+(T) : \; \mu \text{ is supported on } A\},$$

where $\mathrm{rac}_A^+(T)$ is interpreted as $\{0\}$ if $A = \emptyset$. It is well known, as the Riesz representation theorem, that $C(T)^* = \mathrm{rca}(T)$ and that

$$\mu \in \mathrm{rca}(T) \text{ and } \int_T y(t)d\mu \ge 0 \;\; \forall y \in C^+(T) \Longrightarrow \mu \in \mathrm{rca}^+(T),$$

where $C^+(T)$ denotes the set of all nonnegative continuous functions on T. For $y \in \mathcal{C}(T)$, let $I(y) = \{t \in T : \; y(t) = 0\}$.

PROPOSITION 4.15. *Let X be a general Banach space, T a compact topological space, $\psi(x,t)$ a continuous function on $X \times T$ such that $\psi_1'(x,t)$ is continuous on $X \times T$, and $g : X \to \mathcal{C}(T)$ defined by $g(x) := \psi(x,\cdot)$ for all $x \in X$. Let $M : \mathcal{C}(T) \rightrightarrows X$ be defined by (4.21) and let $\bar{x} \in M(0)$. Suppose that there exist $\eta, \delta \in (0, +\infty)$ such that for all $u \in M(0) \cap B(\bar{x}, \delta)$,*

(4.22)

$$N_c(M(0), u) \cap B_{X^*} \subset [0,\ \eta] \left\{ \int_T \psi_1'(u,t)d\mu : \ \mu \in \mathrm{rac}^+_{I(g(u))}(T) \text{ and } \mu(T) \le 1 \right\}.$$

Then M is calm at $(0, \bar{x})$.

Proof. By the assumption on ψ, it is easy to verify that $g'(x) = \psi_1'(x,\cdot)$ for all $x \in X$, and so

$$(g'(x))^*(\mu) = \int_T \psi_1'(x,t)d\mu \quad \forall \mu \in \mathrm{rac}(T) = \mathcal{C}(T)^*.$$

By Theorem 4.14 (applied to $Y = \mathcal{C}(T)$, $\Lambda = -\mathcal{C}^+(T)$, $g(x) = \psi(x,\cdot)$ for all $x \in X$ and $\bar{y} = 0$), we need only show that

$$N(-\mathcal{C}^+(T), -y) = \mathrm{rac}^+_{I(y)}(T) \quad \forall y \in \mathcal{C}^+(T). \tag{4.23}$$

Let $y \in \mathcal{C}^+(T)$ and $\mu \in N(-\mathcal{C}^+(T), -y)$. Since $-\mathcal{C}^+(T)$ is a closed convex cone in $\mathcal{C}(T)$, the Riesz representation theorem implies that

$$\int_T -z(t)d\mu \le \int_T -y(t)d\mu = 0 \quad \forall z \in \mathcal{C}^+(T).$$

It follows that $\mu \in \mathrm{rac}^+(T)$ and $\int_T y(t)d\mu = 0$. This shows that $\mu \in \mathrm{rac}^+_{I(y)}(T)$. Hence, $N(-\mathcal{C}(T), -y) \subset \mathrm{rac}^+_{I(y)}(T)$. Since the reverse inclusion is clear, (4.23) holds. The proof is complete.

Remark 4.4. Note that in the special case when $X = R^n$ and under the assumption of Proposition 4.15,

$$\mathrm{co}\{\psi_1'(u,t) : \ t \in I(g(u))\} = \mathrm{cl}(\mathrm{co}\{\psi_1'(u,t) : \ t \in I(g(u))\})$$

and so

$$\mathrm{co}\{\psi_1'(u,t) : \ t \in I(g(u))\} = \left\{ \int_T \psi_1'(u,t)d\mu : \ \mu \in \mathrm{rac}^+_{I(g(u))}(T) \text{ and } \mu(T) \le 1 \right\}.$$

Thus, (4.22) and (WC) are the same. Hence Proposition 4.15 improves and generalizes Theorem A by Henrion and Outrata. Moreover, Proposition 4.15 does not require that $\psi_1'(x,t)$ is locally Lipschitzian.

5. Application to error bounds for inequality systems. Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function and consider the following inequality system:

$$f(x) \le 0. \tag{5.1}$$

Let $f_1, \dots, f_n : X \to R \cup \{+\infty\}$ be proper lower semicontinuous functions and $f(x) = \max\{f_i(x) : \ i = 1, \dots, n\}$. Then (5.1) is the following system of finitely many inequalities:

$$f_i(x) \le 0, \ \ i = 1, \dots, n. \tag{5.2}$$

Recall that inequality (5.1) has a local error bound (or metric regularity) at $\bar{x}$ if there exists $\tau > 0$ such that

$$d(x, S) \leq \tau[f(x)]_+ \quad \forall x \text{ close to } \bar{x}, \tag{5.3}$$

where S is the solution set of (5.1) and $[f(x)]_+ = \max\{0, f(x)\}$.

In the case when f (resp., f_i) is convex, many authors studied the error bound issues for (5.1) (resp., (5.2)) (see [12, 13, 18, 19, 21, 38, 39, 41] and the references therein). In particular, in the case when f is convex, it is known (cf. [12, 13, 41]) that (5.1) has a local error bound at a point a of the solution set S if and only if there exist $\tau, \delta \in (0, +\infty)$ such that

$$N(S, z) \cap B_{X^*} \subset [0, \tau]\partial f(z) \quad \forall\, z \in \mathrm{bd}(S) \cap B(a, \delta).$$

Under the condition that X is finite-dimensional and each f_i is convex and smooth, Li [19] proved that inequality system (5.2) has a local error bound at $a \in S$ if and only if

$$N(S, z) = R_+ \mathrm{co}\{f_i'(z) : \ i \in I(z)\} \ \ \forall z \in \mathrm{bd}(S) \text{ close to } a,$$

where $I(z) := \{1 \leq i \leq n : \ f_i(z) = 0\}$.

As applications of the main results obtained in section 4, we consider local error bounds for (5.1) and (5.2) when f and f_i are not necessarily convex. For the sake of simplicity in presentation, let us assume, in the remainder of this section, that $f : X \to R$ is a local Lipschitz (not necessarily convex) function.

As an extension of the convexity, Ngai, Luc, and Thera [29] introduced the approximate convexity. Recall that a function $f : X \to R$ is said to be approximately convex at $a \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) + \varepsilon t(1-t)\|x_1 - x_2\|$$

for all $x_1, x_2 \in B(a, \delta)$ and $t \in (0,\ 1)$. Recently, Aussel, Daniilidis, and Thibault [1] proved that a local Lipschitz function $f : X \to R$ is approximately convex at a if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) - f(u) - \langle u^*, x - u\rangle \geq -\varepsilon\|x - u\| \quad \forall x, u \in B(a, \delta) \text{ and } \forall u^* \in \partial_c f(u). \tag{$*$}$$

Slightly weakened conditions can be introduced as follows: f is said to be L-subsmooth (resp., weak L-subsmooth) at $a \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) - f(u) - \langle u^*, x - u\rangle \geq -\varepsilon\|x - u\|$$

whenever $x \in B(u, \delta)$ and $u \in B(a, \delta)$ with $f(u) = f(a)$, $u^* \in \partial_c f(u)$ (resp., $u^* \in \partial f(u)$).

Let $M : R \rightrightarrows X$ be defined by

$$M(y) := \{x \in X : \ f(x) \leq y\} \quad \forall y \in R.$$

Then $\mathrm{Gr}(M^{-1}) = \mathrm{epi}(f)$. Hence,

$$\mathrm{dom}(D_c^* M^{-1}(x, f(x))) = [0,\ +\infty) \ \text{ and } \ D_c^* M^{-1}(x, f(x))(r) = r\partial_c f(x) \tag{5.4}$$

for all $x \in X$ and $r \in [0,\ +\infty)$. Note that $N_c(\mathrm{Gr}(M), (t, x)) = \{(0, 0)\}$ for any $x \in X$ and $t > f(x)$ (because $(x, t) \in \mathrm{int}(\mathrm{Gr}(M^{-1}))$). By the local Lipschitz property of

f, it is easy to verify that f is (weak) L-subsmooth at a if and only if M is (weak) L-subsmooth at $(f(a), a)$. Note that $M(0) = S$, and (5.1) has a local error bound at $a \in S$ if and only if M is calm at $(0, a)$. Thus, the following result is immediate from Theorems 4.2 and 4.4.

THEOREM 5.1. *The following assertions hold.*

(i) *If inequality* (5.1) *has a local error bound at* $a \in S$, *then there exist* $\tau, \delta \in (0, +\infty)$ *such that*

$$\hat{N}(S, x) \cap B_{X^*} \subset [0, \tau]\partial_c f(x) \quad \forall x \in S \cap B(a, \delta).$$

(ii) *If* f *is L-subsmooth at* $a \in S$ *and there exist* $\tau, \delta \in (0, +\infty)$ *such that*

$$N_c(S, x) \cap B_{X^*} \subset [0, \tau]\partial_c f(x) \quad \forall x \in \mathrm{bd}(S) \cap B(a, \delta), \tag{5.5}$$

then (5.1) *has a local error bound at* a.

By the same argument but using Theorems 4.3 and 4.8 (in place of Theorems 4.2 and 4.4), we have the following characterization of a local error bound for inequality (5.1) when X is an Asplund space.

THEOREM 5.2. *Suppose that* X *is an Asplund space and that* f *is weakly L-subsmooth at* $a \in S$. *Then inequality* (5.1) *has a local error bound at* $a \in S$ *if and only if there exist* $\tau, \delta \in (0, +\infty)$ *such that*

$$\hat{N}(S, x) \cap B_{X^*} \subset [0, \tau]\partial f(x) \quad \forall x \in \mathrm{bd}(S) \cap B(a, \delta).$$

The next two theorems (Theorems 5.3 and 5.6) concern convex-composite functions.

THEOREM 5.3. *Let* $\phi : Z \to R$ *be a continuous convex function and* $g : X \to Z$ *be a smooth function. Let* $f(x) = \phi(g(x))$ *for all* $x \in X$. *Let* $a \in S$ *and suppose that* $g'(a)$ *is surjective. Then* (5.1) *has a local error bound at* a *if and only if there exist* $\tau, \delta \in (0, +\infty)$ *such that* (5.5) *holds.*

Proof. Let $G : R \rightrightarrows Z$ and $M : R \rightrightarrows X$ be defined by

$$M(y) := \{x \in X : \ f(x) \le y\} \ \text{ and } \ G(y) := \{z \in Z : \ \phi(z) \le y\} \quad \forall y \in R.$$

Then $M(y) = g^{-1}(G(y))$ for all $y \in R$. It follows from Theorem 4.11 and (5.4) that M is calm at $(0, a)$ if and only if that there exist $\tau, \delta \in (0, +\infty)$ such that (5.5) holds. Since M is calm at $(0, a)$ if and only if (5.1) has a local error bound, the proof is complete.

Theorems 5.1–5.3 can be regarded as generalizations of the main result in [41] from the convex case to the nonconvex case. Next we consider local error bounds for inequality system (5.2).

PROPOSITION 5.4. *Let* $f_1, \dots, f_n : X \to R$ *be smooth (not necessarily convex) functions. Let* $a \in S := \{x \in X : \ f_i(x) \le 0, \ i = 1, \dots, n\}$ *and suppose that there exists* $\tau \in (0, +\infty)$ *such that*

$$N_c(S, z) \cap B_{X^*} \subset [0, \tau]\mathrm{co}(\{f_i'(z) : \ i \in I(z)\}) \ \forall \ z \in \mathrm{bd}(S) \textit{ close to } a.$$

Then (5.2) *has a local error bound at* a.

Proof. Let $f(x) = \max\{f_i(x) : \ i = 1, \dots, n\}$ for all $x \in X$. Then, by [4, Proposition 2.3.12], one has

$$\partial_c f(u) = \mathrm{co}(\{f_i'(u) : \ i \in I(u)\}) \quad \forall u \in X.$$

By (ii) of Theorem 5.1, we need only show that f is L-subsmooth at a. Since each f_i is smooth on X, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f_i(x_1) - f_i(x_2) - \langle f_i'(x_2), x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(a, \delta). \tag{5.6}$$

Let $x, u \in B(a, \delta)$ and $u^* \in \partial_c f(u)$. Then there exist $t_i \geq 0$ $(i \in I(u))$ such that $\sum_{i \in I(u)} t_i = 1$ and $u^* = \sum_{i \in I(u)} t_i f_i'(u)$. Hence, it follows from (5.6) that

$$\begin{aligned} f(x) - f(u) - \langle u^*, x - u\rangle &= \sum_{i \in I(u)} t_i(f(x) - f_i(u) - \langle f_i'(u), x - u\rangle) \\ &\geq -\varepsilon\|x - u\|. \end{aligned}$$

This shows that f is approximately convex (and thus L-subsmooth) at a. The proof is complete.

Now we extend Li's result on local error bounds (i.e., the metric regularity) for a system of smooth and convex inequalities to the nonconvex case. First we prove a lemma.

LEMMA 5.5. *Let $f_1, \ldots, f_n : X \to R$ be smooth functions. Let $a \in \mathrm{bd}(S)$ be such that for any $J \subset I(a)$,*

$$0 \in \mathrm{co}\{f_i'(a) : \ i \in J\} \Rightarrow a \text{ is a local minimizer of } \max\{f_i(x) : \ i \in J\}. \tag{5.7}$$

Then there exists $\tau \in (0, +\infty)$ such that

$$N_c(S, z) \cap B_{X^*} \subset [0, \ \tau]\mathrm{co}\{f_i'(z) : \ i \in I(z)\} \ \ \forall z \in \mathrm{bd}(S) \text{ close to } a \tag{5.8}$$

if and only if

$$N_c(S, z) = R_+\mathrm{co}(\{f_i'(z) : \ i \in I(z)\}) \ \ \forall z \in \mathrm{bd}(S) \text{ close to } a. \tag{5.9}$$

The corresponding result also holds if $N_c(S, z)$ is replaced with $\hat{N}(S, z)$ in (5.8) *and* (5.9).

Proof. We prove only the first assertion (the proof for the last assertion is similar). Since each f_i is smooth, it is easy to verify that

$$R_+\mathrm{co}(\{f_i'(z) : \ i \in I(z)\}) \subset N_c(S, z) \quad \forall z \in \mathrm{bd}(S).$$

We need only show that (5.9) implies that there exists $\tau \in (0, +\infty)$ such that (5.8) holds. To do this, suppose to the contrary that there exist a sequence $\{z_k\}$ in $\mathrm{bd}(S)$ and a sequence $\{z_k^*\}$ in X^* such that

$$z_k \to a \ \text{ and } \ z_k^* \in N_c(S, z_k) \cap B_{X^*} \setminus [0, \ k]\mathrm{co}\{f_i'(z_k) : \ i \in I(z_k)\}. \tag{5.10}$$

By the continuity of f_i and by considering k large if necessary, we assume without loss of generality that $I(z_k) \subset I(a)$ for each k. Similarly, by (5.9), we may assume that for each k there exist $t_k(i) > 0$ $(i \in I(z_k))$ such that $z_k^* = \sum_{i \in I(z_k)} t_k(i) f_i'(z_k)$. It follows from the Carathéodory theorem (cf. [3, p. 25]) that there exist $J_k \subset I(z_k)$ and $r_k(i) > 0$ such that $\{f_i'(z_k) : \ i \in J_k\}$ is linearly independent and $z_k^* = \sum_{i \in J_k} r_k(i) f_i'(z_k)$. This and (5.10) imply that $\sum_{i \in J_k} r_k(i) > k$. Noting that $J_k \subset \{1, \ldots, n\}$, without loss of generality we can assume that $J_k = J$ for each k and

$$\frac{r_k(i)}{\sum_{j \in J} r_k(j)} \to r_i, \ \text{ as } k \to \infty \text{ and } i \in J$$

(passing to a subsequence if necessary). Then $J \subset I(z_k) \subset I(a)$ and $\sum_{i\in J} r_i = 1$. Since $z_k^* \in B_{X^*}$, it follows that

$$0 = \lim_{k\to\infty} \frac{z_k^*}{\sum_{j\in J} r_k(j)} = \lim_{k\to\infty} \frac{\sum_{i\in J} r_k(i) f_i'(z_k)}{\sum_{j\in J} r_k(j)} = \sum_{i\in J} r_i f_i'(a) \in \mathrm{co}\{f_i'(a) :\ i \in J\}.$$

This and (5.7) imply that a is a local minimizer of f_J defined by $f_J(x) := \max\{f_i(x) :\ i \in J\}$. Thus there exists an open neighborhood U of a such that $f_J(x) \geq f_J(a)$ for all $x \in U$. Noting that $f_J(a) = f_J(z_k)$ (as $I(z_k) \subset I(a)$), it follows from (5.10) that z_k is also a local minimizer of f_J for each k large enough. Thus, for each k large enough, $0 \in \mathrm{co}\{f_i'(z_k) :\ i \in J\}$, contradicting the fact that $\{f_i'(z_k) :\ i \in J\}$ is linearly independent.

The following theorem clearly improves and extends Li's result (from the convex and finite-dimensional case to the nonconvex and infinite-dimensional case).

THEOREM 5.6. *Let $f_i(x) = \phi_i(g(x))$ for all $x \in X$ $(i = 1, \ldots, n)$, where $g : X \to Z$ is a smooth mapping, $\phi_i : Z \to R$ is a smooth convex function, and Z is another Banach space. Let $a \in \mathrm{bd}(S)$. Suppose that $g'(a)$ is surjective. Then* (5.2) *has a local error bound at a if and only if*

$$N_c(S, z) = R_+\mathrm{co}(\{f_i'(z) :\ i \in I(z)\}) \text{ for } z \in \mathrm{bd}(S) \text{ close to } a.$$

Proof. Note that (5.2) has a local error bound at a if and only if (5.1) also does with $f(x) = \max\{f_i(x) :\ i = 1, \ldots, n\}$ for all $x \in X$, and also note that

$$\partial_c f(x) = \mathrm{co}(\{f_i'(x) :\ i \in I(x)\}) \quad \forall x \in X.$$

By Theorem 5.3 and Lemma 5.5, we need only show that (5.7) holds. To do this, let $J \subset I(a)$ and $0 \in \mathrm{co}\{f_i'(a) :\ i \in J\}$. Then there exist $\lambda_i \geq 0$ with $\sum_{i\in J} \lambda_i = 1$ such that

$$0 = \sum_{i\in J} \lambda_i f_i'(a) = \sum_{i\in J} \lambda_i [g'(a)]^*(\phi_i'(g(a))) = [g'(a)]^* \left(\sum_{i\in J} \lambda_i \phi_i'(g(a)) \right).$$

Noting that $[g'(a)]^*$ is injective (because $g'(a)$ is surjective), it follows that

$$0 = \sum_{i\in J} \lambda_i \phi_i'(g(a)). \tag{5.11}$$

Let $\phi(u) := \max\{\phi_i(u) :\ i \in J\}$ for all $u \in Z$. Then ϕ is a continuous convex function and, by $J \subset I(a)$, $\phi(g(a)) = \phi_i(g(a))$ for all $i \in J$. This and (5.11) imply that $g(a)$ is a global minimizer of ϕ. It follows that a is a global minimizer of $\max\{f_i(x) :\ i \in J\}$. This shows that (5.7) holds.

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