Nonlinear Analysis 120 (2015) 186–201

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Hölder weak sharp minimizers and Hölder tilt-stability[☆]

Xi Yin Zheng[a,*], Kung-Fu Ng[b]

[a] *Department of Mathematics, Yunnan University, Kunming 650091, PR China*
[b] *Department of Mathematics (and IMS), Chinese University of Hong Kong, Hong Kong, China*

ARTICLE INFO

Article history:
Received 20 September 2014
Accepted 28 February 2015
Communicated by S. Carl

MSC:
90C31
49K40
49J52

Keywords:
Hölder weak sharp minimizer
Stable Hölder weak sharp minimizer
Hölder tilt-stable minimizer
Hölder metric regularity

ABSTRACT

In this paper, we introduce and study the notions of Hölder weak sharp minimizers, stable Hölder weak sharp minimizers and Hölder tilt-stable weak minimizers for a proper lower semicontinuous function f on a Banach space. In terms of the Hölder metric subregularity/regularity of ∂f, we consider optimality conditions for Hölder weak sharp minimizers and stable Hölder weak sharp minimizers. We prove that $\bar{x}$ is a stable Hölder weak sharp minimizer (resp. a Hölder tilt-stable weak minimizer) of f if and only if it is a stable Hölder sharp minimizer (resp. a Hölder tilt-stable minimizer) of f.

1. Introduction

Let X be a Banach space, and we consider a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ (with the effective domain and the Clarke–Rockafellar subdifferential denoted by $\mathrm{dom}(f)$ and ∂f; see the next section for definitions and notations). Recall (cf. [3,7]) that $\bar{x} \in \mathrm{dom}(f)$ is a sharp minimizer (or strong local minimizer) of f if there exist positive constants κ and δ such that

$$\kappa\|x - \bar{x}\| \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta), \tag{1.1}$$

where $B_X(\bar{x}, \delta)$ denotes the open ball of X with center $\bar{x}$ and radius δ (and $B_X[\bar{x}, \delta]$ will be used to denote the corresponding closed ball). The notion of sharp minimizers has been recognized to be useful in convergence analysis of algorithms in optimization. However, the sharp minimizer notion in the sense of (1.1) is a rather strong condition: for example, it can be shown easily that a smooth function does not have any sharp minimizer. Replacing $\|x - \bar{x}\|$ in (1.1) by $\|x - \bar{x}\|^q$ with some constant $q > 1$, one can consider the following

☆ This research was supported by the National Natural Science Foundation of P. R. China (Grant No. 11371312 and 11261067), Earmarked Grants (GRF) from the Research Grant Council of Hong Kong (Project nos. CUHK 402612 and 14304014) and IRTSTY.

* Corresponding author.
E-mail addresses: xyzheng@ynu.edu.cn (X.Y. Zheng), kfng@math.cuhk.edu.hk (K.-F. Ng).

http://dx.doi.org/10.1016/j.na.2015.02.020

weaker notion of a q-order (sharp) minimizer of f: there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\kappa\|x - \bar{x}\|^q \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta). \tag{1.2}$$

In the case of $q = 2$, the 2-order (sharp) minimizer notion in the sense of (1.2) is well-known and has played an important role in perturbation theory and convergence analysis in optimization. Recently, many authors studied the stable 2-order minimizers when the function f undergoes small linear perturbations by considering the functions

$$f_{u^*} := f - u^* \tag{1.3}$$

with u^* in X^* (cf. [2,5,6,12,15,17,16,18,20,21]). Replacing 2 by general q in $(1, +\infty)$, Zheng and Ng [28] further introduced the stable Hölder sharp minimizer: a point $\bar{x} \in \mathrm{dom}(f)$ is said to be a stable q-order (sharp) minimizer of f if there exist $\delta, r, \kappa \in (0, +\infty)$ such that for each $u^* \in B_{X^*}(0, \delta)$ there exists $x_{u^*} \in B_X(\bar{x}, r)$, with $x_0 = \bar{x}$, satisfying the following property:

$$\kappa\|x - x_{u^*}\|^q \le f_{u^*}(x) - f_{u^*}(x_{u^*}) \quad \forall x \in B_X(\bar{x}, r). \tag{1.4}$$

Motivated by the tilt-stability of Poliquin and Rockafellar (see [21]), Zheng and Ng [28] also introduced the following notion: $\bar{x}$ is said to be a tilt-stable p-order minimizer of f (or say that $\bar{x}$ gives a tilt-stable p-order minimum of f) with $p \in (0, +\infty)$ if there exist $r, \delta, L \in (0,\ \infty)$ and $M : B_{X^*}(0, \delta) \to B_X[\bar{x}, r]$ with $M(0) = \bar{x}$ such that

$$f_{u^*}(M(u^*)) = \min_{x \in B_X[\bar{x}, r]} f_{u^*}(x) \quad \forall u^* \in B_{X^*}(0, \delta) \tag{1.5}$$

(where f_{u^*} is as in (1.3)) and

$$\|M(x^*) - M(u^*)\| \le L\|x^* - u^*\|^p \quad \forall x^*, u^* \in B_{X^*}(0, \delta). \tag{1.6}$$

Significant advances have been made regarding the stable minimizers and the tilt-stable sharp minimizers for the case when $q = 2$ and $p = 1$ (cf. [1,2,5,6,12,15,16,18,20,21]). In particular, under the assumption that f is a proper lower semicontinuous function on a Hilbert space and $\bar{x}$ is a local minimizer of f such that f is subdifferentially continuous and proximally regular at $(\bar{x}, 0)$, the following statements are known to be equivalent:

(i) $0 \in \partial f(\bar{x})$ and the generalized second order subdifferential $\partial^2 f(\bar{x}, 0)$ (whose graph is the Mordukhovich normal cone of the graph of ∂f to $(\bar{x}, 0)$) is positively definite.
(ii) the subdifferential mapping ∂f is strongly metrically regular at $\bar{x}$ for 0.
(iii) $\bar{x}$ is a stable 2-order (sharp) minimizer of f.
(iv) $\bar{x}$ is a tilt-stable (1-order) minimizer of f.

Note that the above (i) has no counterpart in the general case when q is any number in $(1, +\infty) \setminus \{2\}$, majorly due to the fact that we do not have a satisfactory notion/theory for the corresponding higher order subdifferentials (especially no 'fractional-order' subdifferentials has been considered). Zheng and Ng [28] extended the mutual equivalences of (ii), (iii) and (iv) to the general case of $q \in (1, +\infty)$.

The q-order (sharp) minimizer in the sense of (1.2) is sufficiently strong to ensure that $\arg\min_{x \in B_X(\bar{x}, \delta)} f = \{\bar{x}\}$ is a singleton. In many cases, it is desirable to consider minimizers which are not necessarily isolated ones such as the weak sharp minimizers considered in the seminal paper by Ferris [7]: $\bar{x}$ is called a (local) weak sharp minimizer of f if there exist $r, \kappa \in (0,\ \infty)$ such that $f(\bar{x}) = \inf_{x \in B_X[\bar{x}, r]} f(x)$ and

$$\kappa d(x, S(f, \bar{x}, r)) \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, r), \tag{1.7}$$

where

$$S(f, \bar{x}, r) := \big\{x \in B_X[\bar{x}, r] :\ f(x) = \inf_{u \in B_X[\bar{x}, r]} f(u)\big\}.$$

The weak sharp minimizer has been well studied and is useful in convergence analysis of algorithm (cf. [3,7,24,25,29] and the references therein). Extending this notion of Ferris, we make the following definition.

Definition 1.1. Let $\bar{x} \in \mathrm{dom}(f)$ and $q \in (1, +\infty)$. We say that $\bar{x}$ is a q-order weak sharp minimizer of f if, in place of (1.7), it holds that

$$\kappa d(x, S(f, \bar{x}, r))^q \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, r). \tag{1.8}$$

Motivated by (1.4), (1.6) and Definition 1.1, we are naturally led to make the following two definitions.

Definition 1.2. Let $\bar{x} \in \mathrm{dom}(f)$ and $q \in (1, +\infty)$. We say that $\bar{x}$ is a stable q-order weak sharp minimizer of f if there exist $\delta, r, \kappa \in (0, +\infty)$ such that $f(\bar{x}) = \inf_{x \in B_X[\bar{x}, r]} f(x)$ and

$$\kappa d(x, S(f_{u^*}, \bar{x}, r))^q \leq f_{u^*}(x) - \min_{x \in B_X[\bar{x}, r]} f_{u^*}(x) \tag{1.9}$$

for all $x \in B_X(\bar{x}, r)$ and $u^* \in B_{X^*}(0, \delta)$, where

$$S(f_{u^*}, \bar{x}, r) := \Big\{u \in B_X[\bar{x}, r] : \ f_{u^*}(u) = \inf_{x \in B_X[\bar{x}, r]} f_{u^*}(x)\Big\}. \tag{1.10}$$

Definition 1.3. Let $\bar{x} \in \mathrm{dom}(f)$ and $p \in (0, +\infty)$. We say that $\bar{x}$ is a tilt-stable weak p-order minimizer of f if there exist $r, \delta, L \in (0, \infty)$ and a neighborhood V of $\bar{x}$ such that $\bar{x} \in S(f, \bar{x}, r)$ and

$$S(f_{x^*}, \bar{x}, r) \cap V \subset S(f_{u^*}, \bar{x}, r) + L\|x^* - u^*\|^p B_{X^*} \quad \forall x^*, u^* \in B_{X^*}(0, \delta). \tag{1.11}$$

Clearly, the reason that the word “weak” appears in Definitions 1.1–1.3 is because we are now in the broader situation that minimizers under consideration are not longer required to restrict to be isolated ones.

The rest of the paper is organized as follows. Section 2 provides some notions and results in variational analysis, which are often used in the sequel. In Section 3, we mainly consider a Hölder weak sharp minimizer for a proper lower semicontinuous function f on a Banach space. In terms of the Hölder metric subregularity of the subdifferential mapping ∂f, we provide some optimality conditions for Hölder weak sharp minimizers of f. In the case when f is a twice smooth function on a Euclidean space, we establish the relationship between the positive definiteness of $f''(\bar{x})$ on the normal cone $N(S(f, \bar{x}, r), \bar{x})$ and the fact that $\bar{x}$ is a 2-order weak sharp minimizer of f. In Section 4, we consider the stable Hölder weak sharp minimizers and the tilt-stable Hölder weak minimizers. Unlike the occurrence of two distinct notions of the Holder weak sharp minimizers and the Holder minimizers, a somewhat surprising result established in Corollary 4.1 is that $\bar{x}$ is a stable q-order weak sharp minimizer (resp. a tilt-stable p-order weak minimizer) of f if and only if it is a stable q-order sharp minimizer (resp. a tilt-stable p-order minimizer) of f.

2. Preliminaries

Let X be a Banach space with the topological dual X^*. For a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, the Clarke–Rockafellar subdifferential $\partial f(\bar{x})$ of f at $\bar{x} \in \mathrm{dom}(f)$ is defined as

$$\partial f(\bar{x}) := \{x^* \in X^* | \langle x^*, h\rangle \leq f^{\uparrow}(\bar{x}, h) \ \forall h \in X\}$$

where

$$f^{\uparrow}(\bar{x}, h) := \lim_{\varepsilon \downarrow 0} \limsup_{x \xrightarrow{f} \bar{x}, t \downarrow 0} \inf_{w \in h + \varepsilon B_X} \frac{f(x + tw) - f(x)}{t}.$$

In the case when f is locally Lipschitzian around $\bar{x}$, $f^{\uparrow}(\bar{x},h)$ reduces to the Clarke directional derivative

$$f^{\circ}(\bar{x},h) := \limsup_{t\to 0^+, x\to\bar{x}} \frac{f(x+th)-f(x)}{t}.$$

It is well known that if f is convex then

$$\partial f(\bar{x}) = \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \le f(x)-f(\bar{x})\, \forall x\in X\}.$$

The following lemmas are well known and fundamental in variational analysis.

Lemma 2.1. *Let X be a Banach space and $f_1, f_2 : X \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous functions. Let $\bar{x} \in \mathrm{dom}(f_1)\cap\mathrm{dom}(f_2)$ be such that f_1 is locally Lipschitzian around $\bar{x}$. Then*

$$\partial(f_1+f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Lemma 2.2. *Let X, Y be Banach spaces, $\phi : Y \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function, $g : X \to Y$ be a smooth function, and let $\bar{x} \in \mathrm{dom}(\phi\circ g)$ be such that the derivative $g'(\bar{x})$ is surjective. Then*

$$\partial(\phi\circ g)(\bar{x}) = g'(\bar{x})^*(\partial f(g(\bar{x}))).$$

We will also need the following lemma (cf. [14, Theorem 1.57] and [26, Lemma 3.6]).

Lemma 2.3. *Let X, Y be Banach spaces and $g : X \to Y$ be a smooth function. Let $\bar{x}\in X$ be such that $\nabla g(\bar{x})$ is surjective. Then there exist $\delta, \tau, L_0, L, M \in (0,+\infty)$ such that*

$$d(x, g^{-1}(y)) \le \tau\|y-g(x)\| \quad \forall (x,y)\in B_X(\bar{x},\delta)\times B_Y(g(\bar{x}),\delta), \tag{2.1}$$

$$\|g(x_2)-g(x_1)\| \le L_0\|x_2-x_1\| \quad \forall x_1,x_2\in B_X(\bar{x},\delta), \tag{2.2}$$

$$LB_Y \subset \nabla g(x)(B_X) \quad \forall x\in B_X(\bar{x},\delta), \tag{2.3}$$

$$L\|y^*\| \le \|\nabla g(x)^*(y^*)\| \le M\|y^*\| \quad \forall (x,y^*)\in B_X(\bar{x},\delta)\times Y^*. \tag{2.4}$$

Let F be a multifunction between two Banach spaces X and Y. For $p\in(0,+\infty)$ and $(\bar{x},\bar{y})\in\mathrm{gph}(F)$, we say that F is

(i) p-order metrically regular at $\bar{x}$ for $\bar{y}$ if there exist $\tau,\delta\in(0,+\infty)$ such that

$$d(x, F^{-1}(y)) \le \tau d(y, F(x))^p \quad \forall (x,y)\in B_X(\bar{x},\delta)\times B_Y(\bar{y},\delta). \tag{2.5}$$

(ii) p-order strongly metrically regular at $\bar{x}\in X$ for $\bar{y}$ if there exist $\tau,\delta,\eta\in(0,+\infty)$ such that (2.5) holds and $F^{-1}(y)\cap B_X(\bar{x},\eta)$ is a singleton for each $y\in B_Y(\bar{y},\delta)$.

(iii) p-order (strongly) metrically subregular at $\bar{x}$ for $\bar{y}$ if there exist $\tau,\delta\in(0,+\infty)$ such that $(F^{-1}(\bar{y})\cap B_X(\bar{x},\delta)=\{\bar{x}\}$ and)

$$d(x, F^{-1}(\bar{y})) \le \tau d(\bar{y}, F(x))^p \quad \forall x\in B_X(\bar{x},\delta). \tag{2.6}$$

In the case of $p=1$, the metric regularity/subregularity has been well studied (cf. [1,4,9,26,27] and the references therein). But, there are only few studies for p-order metric regularity/subregularity with $p\in(0,+\infty)\setminus\{1\}$ (cf. [8,11,13]).

Finally, we recall the notion of the limiting normal cone defined by Mordukhovich. For a closed subset A of $\mathbb{R}^n$ and $\bar{x} \in A$, let $N^P(A, \bar{x})$ denote the proximal normal cone of A to $\bar{x}$, namely $h \in N^P(A, \bar{x})$ if and only if

$$t\|h\| = d(\bar{x} + th, A) \quad \text{for all sufficiently small } t > 0 \text{ (cf. [22])}.$$

The limiting normal cone of A to $\bar{x}$ is denoted by

$$N(A, \bar{x}) := \{h \in \mathbb{R}^n : \exists x_k \to \bar{x} \text{ and } h_k \to h \text{ such that } h_k \in N^P(A, x_k)\}.$$

3. Hölder weak sharp minimizer

In this section, in terms of Hölder metric subregularity of the subdifferential mapping ∂f, we mainly consider the Hölder weak sharp minimizers for a proper lower semicontinuous function f.

The following result is inspired by [28, Theorem 4.1] (and [23, Theorems 3.1 and 3.1']). To the best knowledge of us, [28] was the first to note the exact quantitative relationship among the Hölder order of the strong metric regularity/subregularity of ∂f at $\bar{x}$ for 0, the Hölder order of the sharp minimizer $\bar{x}$ of f and the Hölder order of the metric regularity of the corresponding subdifferential mapping even though Hölder metric regularity has been studied by many authors, while somewhat earlier [23] considered the more generalized metric subregularity for the subdifferential mapping with respect to an admissible function.

Theorem 3.1. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $p \in (0, +\infty), r \in (0, +\infty]$ and let $\bar{x} \in A \subset (\partial f)^{-1}(0)$. Then the following statements hold:*

(i) *Suppose that there exist $\kappa, \delta \in (0, +\infty)$ such that*

$$\kappa d(x, A) \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta), \tag{3.1}$$

and let τ and η be positive constants defined by

$$\tau := \frac{p\kappa^{\frac{1}{p}}}{(1+p)^{\frac{1+p}{p}}} \quad \text{and} \quad \eta := \frac{1+p}{1+2p}\min\{r, \delta\}. \tag{3.2}$$

Then

$$\tau d(x, A)^{\frac{1+p}{p}} \le f(x) - \inf_{u \in B_X[\bar{x}, r]} f(u) \quad \forall x \in B_X(\bar{x}, \eta). \tag{3.3}$$

(ii) *Suppose that f is convex and that there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\tau d(x, A)^{\frac{1+p}{p}} \le f(x) - \inf_{u \in B_X[\bar{x}, r]} f(u) \quad \forall x \in B_X(\bar{x}, \delta). \tag{3.4}$$

Then

$$\tau^p d(x, A) \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta). \tag{3.5}$$

Consequently, under the convexity assumption on $f, \bar{x}$ is a $\frac{1+p}{p}$-order weak sharp minimizer of f if and only if ∂f is p-order metrically subregular at $\bar{x}$ for 0.

Proof. (i) Suppose to the contrary that (3.3) is not true, namely there exists $x_0 \in B(\bar{x}, \eta)$ such that

$$f(x_0) < \inf_{x \in B_X[\bar{x}, r]} f(x) + \tau d(x_0, A)^{\frac{1+p}{p}}.$$

Take a $\tau' \in (0,\ \tau)$ sufficiently close to τ such that

$$f(x_0) < \inf_{x \in B_X[\bar{x}, r]} f(x) + \tau' d(x_0, A)^{\frac{1+p}{p}}.$$

Then, by the Ekeland variational principle, there exists $z \in B_X[\bar{x}, r]$ such that

$$\|z - x_0\| < \frac{p}{1+p} d(x_0, A) \tag{3.6}$$

and

$$f(z) \le f(x) + \frac{(1+p)\tau' d(x_0, A)^{\frac{1}{p}}}{p} \|x - z\| \quad \forall x \in B_X[\bar{x}, r]. \tag{3.7}$$

Note, by (3.6), that

$$\frac{d(x_0, A)}{1+p} \le d(z, A) \tag{3.8}$$

(as $d(x_0, A) - d(z, A) \le \|z - x_0\|$) and that

$$\|z - \bar{x}\| \le \|z - x_0\| + \|x_0 - \bar{x}\| \le \frac{1+2p}{1+p} \|x_0 - \bar{x}\| < \frac{(1+2p)\eta}{1+p}$$

(because $\bar{x} \in A$ and $x_0 \in B_X(\bar{x}, \eta)$). By the definition of η, it follows that $z \in B_X(\bar{x}, r) \cap B_X(\bar{x}, \delta)$. Hence, by (3.1) and (3.7), we have

$$\kappa d(z, A) \le d(0, \partial f(z))^p$$

and $0 \in \partial \left(f + \frac{(1+p)\tau' d(x_0, A)^{\frac{1}{p}}}{p} \| \cdot - z\| \right)(z)$ which implies by Lemma 2.1 that

$$0 \in \partial f(z) + \frac{(1+p)\tau' d(x_0, A)^{\frac{1}{p}}}{p} B_{X^*}.$$

Consequently it follows from (3.8) that

$$\frac{\kappa d(x_0, A)}{1+p} \le d(0, \partial f(z))^p \le \left(\frac{(1+p)\tau'}{p} \right)^p d(x_0, A)$$

and so $\frac{\kappa}{1+p} \le \left(\frac{(1+p)\tau'}{p} \right)^p$, that is $\tau = \frac{p\kappa^{\frac{1}{p}}}{(1+p)^{\frac{1+p}{p}}} \le \tau'$ (by (3.2)), which contradicts the choice $\tau' < \tau$.

(ii) Let $x \in B_X(\bar{x}, \delta)$ and take a sequence $\{a_n\}$ in A such that

$$d(x, A) = \lim_{n \to \infty} \|x - a_n\|.$$

Since $\bar{x} \in A \subset (\partial f)^{-1}(0)$, it follows from the convexity of f that $f(a_n) = f(\bar{x}) = \inf_{u \in X} f(u)$ for all $n \in \mathbb{N}$. Let $x^* \in \partial f(x)$. Then

$$f(x) - f(\bar{x}) = f(x) - f(a_n) \le \langle x^*, x - a_n \rangle \le \|x^*\| \, \|x - a_n\| \quad \forall n \in \mathbb{N}.$$

This and (3.4) imply that

$$\tau d(x, A)^{\frac{1+p}{p}} \le \|x^*\| \, \|x - a_n\| \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, one has $\tau d(x, A)^{\frac{1}{p}} \le \|x^*\|$. Since x^* is arbitrary in $\partial f(x)$, we have that (3.5) holds. The proof is complete.

In the case when $X = A$, the part (i)⇒(ii) of [19, Theorem 3.4] is similar to Theorem 3.1, which replaces our $\inf_{x\in B(\bar{x},r)} f(x)$ with $f(\bar{x})$ but needs a additional condition ((3.4) in [19]). Moreover, (ii) of [19, Theorem 3.4] says that there are two positive numbers α and η such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x}\rangle + \frac{q\alpha}{1+q} d(x, (\partial f)^{-1}(\bar{x}^*))^{\frac{1+q}{q}} \quad \forall x \in B(\bar{x}, \eta).$$

Thus, their modulus $\frac{q\alpha}{1+q}$ and radius η are only of the existence, while our modulus τ and radius η have exact quantitative relation with the modulus κ and radius δ in (3.1), which is important in the proofs of the later main theorems.

If one drops the convexity assumption, the characterization given in Theorem 3.1 is not valid (see an example given at the end of this section). Nevertheless, the characterization does remain to hold if f is a composition function of the form $f := \phi \circ g$, where $\phi : Y \to \overline{\mathbb{R}}$ is a proper lower semicontinuous convex function and $g : X \to Y$ is a smooth function such that $\nabla g(\bar{x})$ is surjective (thus, by Lemma 2.3, (2.1)–(2.4) hold with some positive constants δ, τ, L_0, L, M).

Theorem 3.2. *Let X, Y be Banach spaces, $\phi : Y \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function, and let $g : X \to Y$ be a smooth function. Define $f : X \to \overline{\mathbb{R}}$ as follows*

$$f(x) := \phi(g(x)) \quad \forall x \in X.$$

Let $p \in (0, +\infty)$ and let $\bar{x} \in \mathrm{dom}(f)$ be a local minimizer of f such that $\nabla g(\bar{x})$ is surjective; explicitly suppose that there exist $r, \delta, \tau, L_0, L, M \in (0, +\infty)$ such that (2.1)–(2.4) hold and

$$f(\bar{x}) = \min_{x\in B_X[\bar{x},r]} f(x). \tag{3.9}$$

Then the following assertions hold:

(i) *If there exist $\kappa_0, \delta_0 \in (0, +\infty)$ such that*

$$\kappa_0 d(x, (\partial f)^{-1}(0)) \leq d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta_0) \tag{3.10}$$

and τ_0 and δ_0' are positive constants defined by

$$\tau_0 := \frac{p\kappa_0^{\frac{1}{p}}}{(1+p)^{\frac{1+p}{p}}} \quad \text{and} \quad \delta_0' := \min\left\{\frac{r}{2}, \frac{\delta}{2}, \frac{(1+p)\delta_0}{1+2p}\right\}, \tag{3.11}$$

then

$$\tau_0 d(x, S(f, \bar{x}, r))^{\frac{1+p}{p}} \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta_0'). \tag{3.12}$$

(ii) *Conversely, if there exist $\tau_1, \delta_1 \in (0, +\infty)$ such that*

$$\tau_1 d(x, S(f, \bar{x}, r))^{\frac{1+p}{p}} \leq f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \delta_1) \tag{3.13}$$

and κ_1 and δ_1' are positive constants defined by

$$\kappa_1 := \left(\frac{\tau_1 L}{L_0}\right)^p \quad \text{and} \quad \delta_1' := \min\left\{r, \frac{\delta}{2}, \delta_1\right\}, \tag{3.14}$$

then

$$\kappa_1 d(x, (\partial f)^{-1}(0)) \le d(0, \partial f(x))^p \quad \forall x \in B_X(\bar{x}, \delta_1'). \tag{3.15}$$

Consequently, $\bar{x}$ is a $\frac{1+p}{p}$-order weak sharp minimizer of f if and only if ∂f is p-order metrically subregular at $\bar{x}$ for 0.

Proof. By (2.3), $\nabla g(x)$ is surjective on $B_X(\bar{x}, \delta)$ and so, by Lemma 2.2, one has

$$\partial f(x) = \nabla g(x)^*(\partial\phi(g(x))) \quad \forall x \in B_X(\bar{x}, \delta). \tag{3.16}$$

Now suppose that there exist $\kappa_0, \delta_0 \in (0, +\infty)$ such that (3.10) holds. Then, by Theorem 3.1(i),

$$\tau_0 d(x, (\partial f)^{-1}(0))^{\frac{1+p}{p}} \le f(x) - f(\bar{x}) \quad \forall x \in B_X(\bar{x}, \eta_0), \tag{3.17}$$

where τ_0 is as in (3.11) and $\eta_0 := \frac{1+p}{1+2p}\min\{r, \delta_0\}$. Let $\gamma := \frac{1}{2}\min\{r, \delta\}$. We claim that

$$(\partial f)^{-1}(0) \cap B_X(\bar{x}, 2\gamma) \subset S(f, \bar{x}, r). \tag{3.18}$$

Granting this, one has

$$d(x, S(f, \bar{x}, r)) \le d(x, (\partial f)^{-1}(0) \cap B_X(\bar{x}, 2\gamma)) = d(x, (\partial f)^{-1}(0)) \quad \forall x \in B_X(\bar{x}, \gamma)$$

(the above equality holds because $\bar{x} \in (\partial f)^{-1}(0)$). Noting that $\delta_0' = \min\{\eta_0, \gamma\}$, it follows from (3.17) that (3.12) holds. Thus, to prove (i), it remains to show that (3.18) holds. Let $u \in (\partial f)^{-1}(0) \cap B_X(\bar{x}, 2\gamma)$. Then $0 \in \partial f(u) = \nabla g(u)^*(\partial\phi(g(u)))$ (thanks to (3.16)). Thus, by (2.4), one has $0 \in \partial\phi(g(u))$. Hence $g(u)$ is a minimizer of the convex function ϕ and so

$$f(u) = \phi(g(u)) \le \phi(g(x')) = f(x') \quad \forall x' \in X.$$

Since $\|u - \bar{x}\| \le 2\gamma < r$, it follows from (3.9) that $u \in S(f, \bar{x}, r)$. Hence (3.18) holds.

To prove (ii), suppose that there exist $\tau_1, \delta_1 \in (0, +\infty)$ such that (3.13) holds. Let $x \in B_X(\bar{x}, \delta_1')$ and $x^* \in \partial f(x)$, where δ_1' is as in (3.14). By (3.16) and (2.4), x^* can be represented as $x^* = \nabla g(x)^*(y^*)$ for some $y^* \in \partial\phi(g(x))$ with $L\|y^*\| \le \|x^*\|$. For (3.15), it suffices to show that

$$\kappa_1 d(x, (\partial f)^{-1}(0)) \le (L\|y^*\|)^p. \tag{3.19}$$

To do this, take a sequence $\{x_n\}$ in $S(f, \bar{x}, r)$ such that

$$\|x - x_n\| \to d(x, S(f, \bar{x}, r)). \tag{3.20}$$

Since $\bar{x} \in S(f, \bar{x}, r)$ and $\|x - \bar{x}\| < \delta_1' \le \frac{\delta}{2}$, we may assume without loss of generality that the sequence $\{x_n\}$ lies in $B(\bar{x}, \delta)$. By (3.13) and the convexity of ϕ, for each $n \in \mathbb{N}$, one has

$$\begin{aligned}
\tau_1 d(x, S(f, \bar{x}, r))^{\frac{1+p}{p}} &\le f(x) - f(\bar{x}) = f(x) - f(x_n) \\
&= \phi(g(x)) - \phi(g(x_n)) \\
&\le \langle y^*, g(x) - g(x_n)\rangle \\
&\le \|y^*\|\, |g(x) - g(x_n)| \\
&\le L_0\|y^*\|\, \|x - x_n\|
\end{aligned}$$

(thanks to (2.2)). Passing to the limit as $n \to \infty$ and making use of (3.20), we have $\tau_1 d(x, S(f, \bar{x}, r))^{\frac{1}{p}} \le L_0\|y^*\|$. Since $S(f, \bar{x}, r) \subset (\partial f)^{-1}(0)$ and $\kappa_1 := \left(\frac{\tau_1 L}{L_0}\right)^p$, this implies that (3.19) holds. The proof is complete.

Dropping the composite-convexity assumption, Theorem 3.2 is not necessarily true. Indeed, for any $p \in (0, +\infty)$, take two sequences $\{a_n\}$ and $\{\varepsilon_n\}$ in $(0,\ 1)$ such that

$$a_n \to 0, \qquad \frac{\varepsilon_n + \varepsilon_n^p}{a_n} \to 0, \quad \text{and} \quad a_n + 2\varepsilon_n < a_{n-1} \quad \forall n \in \mathbb{N}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \le 0 \\ a_n + (2 - \varepsilon_n)(x - a_n), & \text{if } a_n < x \le a_n + \varepsilon_n\ (n \in \mathbb{N}) \\ a_n + 2\varepsilon_n - \varepsilon_n^2 + \varepsilon_n(x - a_n - \varepsilon_n), & \text{if } a_n + \varepsilon_n < x \le a_n + 2\varepsilon_n\ (n \in \mathbb{N}) \\ x, & \text{if } a_n + 2\varepsilon_n < x \le a_{n-1}\ (n \in \mathbb{N}) \\ x, & \text{if } a_0 < x. \end{cases}$$

Then, f is a nonnegative Lipschitz function, $x \le f(x)$ for all $x \in [0, +\infty)$, and

$$S(f, 0) := \{u \in \mathbb{R} :\ f(u) = \min_{x \in \mathbb{R}} f(x)\} = (-\infty,\ 0].$$

Hence

$$d(x, S(f, 0))^{\frac{1+p}{p}} = x^{\frac{1+p}{p}} \le f(x) - f(0) \quad \forall x \in (0,\ 1].$$

This implies that 0 is a $\frac{1+p}{p}$-order weak sharp minimizer of f. On the other hand, it is easy to verify that

$$\partial f(x) = \begin{cases} \{0\}, & \text{if } x \in (-\infty,\ 0) \\ [0,\ 2], & \text{if } x = 0 \\ [1,\ 2 - \varepsilon_n], & \text{if } x = a_n\ (n \in \mathbb{N}) \\ \{2 - \varepsilon_n\}, & \text{if } a_n < x < a_n + \varepsilon_n\ (n \in \mathbb{N}) \\ [\varepsilon_n,\ 2 - \varepsilon_n], & \text{if } x = a_n + \varepsilon_n\ (n \in \mathbb{N}) \\ \{\varepsilon_n\}, & \text{if } a_n + \varepsilon_n < x < a_n + 2\varepsilon_n\ (n \in \mathbb{N}) \\ [\varepsilon_n,\ 1], & \text{if } x = a_n + 2\varepsilon_n\ (n \in \mathbb{N}) \\ \{1\}, & \text{if } x \in \bigcup_{n=1}^{\infty} \big(a_n + 2\varepsilon_n,\ a_{n-1}\big) \cup [a_0, +\infty). \end{cases}$$

Hence $(\partial f)^{-1}(0) = (-\infty,\ 0]$ and so

$$\lim_{n\to\infty} \frac{d(0, \partial f(a_n + \varepsilon_n))^p}{d(a_n + \varepsilon_n, (\partial f)^{-1}(0))} = \lim_{n\to\infty} \frac{\varepsilon_n^p}{a_n + \varepsilon_n} = 0.$$

This implies that ∂f is not p-order metrically subregular at 0 for 0.

Let f be a proper lower semicontinuous function on $\mathbb{R}^n$ and let $\bar{x} \in \mathrm{dom}(f)$ be a local minimizer of f such that f is twice smooth around $\bar{x}$. It is well known that the following statements are equivalent:

(i) $\bar{x}$ is a 2-order sharp minimizer of f.
(ii) $\bar{x}$ is a stable 2-order sharp minimizer of f.
(iii) $f''(\bar{x})$ is positive definite (i.e., $0 < f''(\bar{x})(h^2)$ for all $h \in \mathbb{R}^n \setminus \{0\}$).

Next we consider the corresponding issue for the 2-order weak sharp minimizer case. To do this, we adopt the following notation:

$$S(f, \bar{x}) := \{x \in X :\ f(x) = f(\bar{x})\}.$$

It is clear that $S(f, \bar{x}, r) = S(f, \bar{x}) \cap B[\bar{x}, r]$ if $f(\bar{x}) = \min_{x \in B[\bar{x}, r]} f(x)$.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function and let $\bar{x} \in \mathrm{dom}(f)$ be a local minimizer of f such that f is twice smooth around $\bar{x}$. Then $\bar{x}$ is a 2-order weak sharp minimizer of f if and only if*

$$0 < f''(\bar{x})(h^2) \quad \forall h \in N(S(f,\bar{x}),\bar{x}) \setminus \{0\}. \tag{3.21}$$

Proof. Since f is twice smooth around $\bar{x}$ and $\bar{x}$ is a local minimizer of f, there exists $r > 0$ such that

$$f(\bar{x}) = \min_{x \in B_{\mathbb{R}^n}[\bar{x},r]} f(x),$$

f is twice smooth on $B_{\mathbb{R}^n}[\bar{x}, r]$ and $S(f, \bar{x}, r) = S(f, \bar{x}) \cap B_{\mathbb{R}^n}[\bar{x}, r]$ is closed.

To prove the sufficiency part, suppose to the contrary that (3.21) holds but there exists a sequence $\{x_k\} \subset \mathbb{R}^n$ convergent to $\bar{x}$ such that

$$\frac{d(x_k, S(f,\bar{x}))^2}{k} > f(x_k) - f(\bar{x}) \quad \forall k \in \mathbb{N}. \tag{3.22}$$

Thus, each x_k does not lie in $S(f, \bar{x})$. For each $k \in \mathbb{N}$, take a $u_k \in S(f, \bar{x}) \cap B_{\mathbb{R}^n}[\bar{x}, r]$ such that $\|x_k - u_k\| = d(x_k, S(f, \bar{x})) \cap B_{\mathbb{R}^n}[\bar{x}, r])$. It is easy to verify that $u_k \to \bar{x}$ and so $\|x_k - u_k\| = d(x_k, S(f, \bar{x}))$ for all $k \in \mathbb{N}$ sufficiently large. Hence $\frac{x_k-u_k}{\|x_k-u_k\|} \in N^P_{S(f,\bar{x})}(u_k)$ for all $k \in \mathbb{N}$ sufficiently large. Without loss of generality, we assume that $\frac{x_k-u_k}{\|x_k-u_k\|} \to h_0$ (passing to a subsequence if necessary). Then

$$h_0 \in N(S(f,\bar{x}),\bar{x}) \setminus \{0\}. \tag{3.23}$$

On the other hand, (3.22) and the Taylor theorem imply that

$$\frac{\|x_k - u_k\|^2}{k} > f(x_k) - f(\bar{x}) = f(x_k) - f(u_k) = f''(u_k + \theta_k(x_k - u_k))((x_k - u_k)^2),$$

where $\theta_k \in (0,\ 1)$. Hence $\frac{1}{k} > f''(u_k + \theta_k(x_k - u_k))\left(\left(\frac{x_k-u_k}{\|x_k-u_k\|}\right)^2\right)$. Letting $k \to \infty$, it follows that $0 \geq f''(\bar{x})(h_0^2)$, contradicting (3.21) and (3.23).

Next suppose that $\bar{x}$ is a 2-order weak sharp minimizer of f: there exist $\tau, \delta \in (0, +\infty)$ such that

$$\tau d(x, S(f,\bar{x},r))^2 \leq f(x) - f(\bar{x}) \quad \forall x \in B_{\mathbb{R}^n}(\bar{x},\delta). \tag{3.24}$$

Let $h \in N(S(f, \bar{x}), \bar{x})$. Then there exist sequences $\{u_k\} \subset S(f, \bar{x})$ and $\{h_k\} \subset \mathbb{R}^n$ such that

$$u_k \to \bar{x}, \qquad h_k \to h \quad \text{and} \quad h_k \in N^P_{S(f,\bar{x})}(u_k) \quad \forall k \in \mathbb{N}.$$

Hence there exists a sequence $\{t_k\} \subset (0, +\infty)$ convergent to 0 such that

$$t_k\|h_k\| = d(u_k + t_k h_k, S(f,\bar{x})) \quad \forall k \in \mathbb{N}.$$

Noting that $d(u_k+t_kh_k, S(f, \bar{x}, r)) = d(u_k+t_kh_k, S(f, \bar{x}))$ for all sufficiently large k (because $u_k+t_kh_k \to \bar{x}$), it follows from (3.24) that

$$\tau t_k^2\|h_k\|^2 \leq f(u_k + t_k h_k) - f(u_k) = \frac{1}{2} f''(u_k + \theta_k t_k h_k)(t_k^2 h_k^2)$$

for all sufficiently large k, where $\theta_k \in (0,\ 1)$. Noting that f'' is continuous at $\bar{x}$, it follows that $\tau\|h\|^2 \leq \frac{1}{2} f''(\bar{x})(h^2)$. This shows that (3.21) holds and hence the necessity part holds. The proof is complete.

In the case when $\bar{x}$ is a local isolated minimizer of f, it is well known and easy to verify that $N(S(f, \bar{x}), \bar{x}) = \mathbb{R}^n$; so (3.21) means that $f''(\bar{x})$ is positively definite (on the entire $\mathbb{R}^n$). Thus, Proposition 3.1 partially extends the above mentioned characterization on 2-order sharp minimizer. However, different from the isolated minimizer case, (3.21) does not imply that $\bar{x}$ is a stable 2-order weak sharp minimizer of f.

Indeed, let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, y) = x^2$ for all $(x, y) \in \mathbb{R}^2$. Clearly, $(0, 0)$ is a global minimizer of $f, S(f, (0, 0)) = \{0\} \times \mathbb{R}, N(S(f, (0, 0)), (0, 0)) = \mathbb{R} \times \{0\}$, and

$$f''(0, 0)(h^2) = 2u^2 > 0 \quad \forall h = (u, 0) \in N(S(f, (0, 0)), (0, 0)) \setminus \{(0, 0)\}.$$

On the other hand, for any $\varepsilon > 0$, let $u_\varepsilon^* = (0, \varepsilon)$; then $f_{u_\varepsilon^*}(x, y) = x^2 - \varepsilon y$ for all $(x, y) \in \mathbb{R}^2$ and $f_{u_\varepsilon^*}$ has no local minimizers. Hence $(0, 0)$ is not a stable 2-order weak sharp minimizer of f.

4. Stable Hölder weak sharp minimizer and stable Hölder sharp minimizer

Regarding the stable Hölder (sharp) minimizers and tilt-stable Hölder minimizers, Zheng and Ng [28] has recently proved the following result.

Theorem SH. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ be a minimizer of f and p be a positive number. Consider the following statements:*

(i) *$\bar{x}$ is a tilt-stable p-order minimizer of f.*
(ii) *$\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f.*
(iii) *∂f is p-order strongly metrically regular at $\bar{x}$ for 0.*
(iv) *f^* is $C^{1,p}$-smooth on some neighborhood of 0.*

Then (i) ⇔ (ii) ⇐ (iii) *always hold. If, in addition, f is convex, then* (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) *are true.* ■

In this section, we consider the corresponding issues regarding weak sharp versions namely in exploring relationship between p-order tilt-stable weak sharp minimizers and the q-order stable weak sharp minimizers for functions f *together with the metric regularity of their subdifferentials.*

Theorem 4.1. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ and $p \in (0, +\infty)$. Suppose that there exist $r, \delta \in (0, +\infty)$ and a multifunction $\mathcal{A} : B_{X^*}(0, \delta) \rightrightarrows B_X[\bar{x}, r]$ with $\bar{x} \in \mathcal{A}(0)$ such that*

$$\mathcal{A}(u^*) \subset S(f_{u^*}, \bar{x}, r) \quad \forall u^* \in B_{X^*}(0, \delta) \tag{4.1}$$

and $\mathcal{A}$ has p-order Aubin property on $B_{X^*}(0, \delta)$: there exist a neighborhood V of $\bar{x}$ and $L \in (0, +\infty)$ such that

$$\mathcal{A}(u^*) \cap V \subset \mathcal{A}(x^*) + L\|u^* - x^*\|^p B_X \quad \forall x^*, u^* \in B_{X^*}(0, \delta), \tag{4.2}$$

where f_{u^*} and $S(f_{u^*}, \bar{x}, r)$ are defined by (1.3) and (1.10), respectively. Then there exists $\delta' \in (0, \delta)$ such that, on $B_{X^*}(0, \delta'), \mathcal{A}$ is single-valued, the conjugate function $(f + \delta_{B_X[\bar{x},r]})^*$ is $C^{1,p}$-smooth and

$$S(f_{u^*}, \bar{x}, r) = \mathcal{A}(u^*) = \{\nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)\} \quad \forall u^* \in B_{X^*}(0, \delta'); \tag{4.3}$$

in particular,

$$\|\nabla(f + \delta_{B_X[\bar{x},r]})^*(x^*) - \nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)\| \le L\|x^* - u^*\|^p \tag{4.4}$$

for all $x^*, u^* \in B_{X^*}(0, \delta')$.

Proof. Let $x^* \in B_{X^*}(0, \delta)$ and $x \in S(f_{x^*}, \bar{x}, r)$. Then, by (1.10),

$$(f + \delta_{B_X[0,r]})^*(x^*) = \langle x^*, x\rangle - f(x). \tag{4.5}$$

This implies that $x \in \partial(f + \delta_{B_X[\bar{x},r]})^*(x^*)$ (cf. [25]). Hence

$$S(f_{x^*}, \bar{x}, r) \subset \partial(f + \delta_{B_X[\bar{x},r]})^*(x^*) \quad \forall x^* \in B_{X^*}(0, \delta). \tag{4.6}$$

Since $\bar{x} \in \mathcal{A}(0) \cap V$, (4.2) implies that

$$\bar{x} \in \mathcal{A}(u^*) + L\|u^*\|B_X \quad \forall u^* \in B_{X^*}(0, \delta)$$

and so there exists $\delta' \in (0,\ \delta)$ such that

$$\mathcal{A}(u^*) \cap V \neq \emptyset \quad \forall u^* \in B_{X^*}(0, \delta').$$

Let $u^* \in B_{X^*}(0, \delta')$ and $u \in \mathcal{A}(u^*) \cap V$. Then, by (4.2), for each $x^* \in B_{X^*}(0, \delta)$ there exists $x \in \mathcal{A}(x^*)$ such that

$$\|u - x\| \leq L\|u^* - x^*\|^p. \tag{4.7}$$

By (4.1) and (4.6), one has

$$u \in \partial(f + \delta_{B_X[0,r]})^*(u^*) \quad \text{and} \quad x \in \partial(f + \delta_{B_X[0,r]})^*(x^*).$$

Since the conjugate function $(f + \delta_{B_X[0,r]})^*$ is convex, it follows that

$$\begin{aligned} 0 &\leq (f + \delta_{B_X[0,r]})^*(x^*) - (f + \delta_{B_X[0,r]})^*(u^*) - \langle u, x^* - u^* \rangle \\ &\leq \langle x - u, x^* - u^* \rangle \leq L\|x^* - u^*\|^{1+p} \end{aligned}$$

where the last inequality holds thanks to (4.7). Hence $(f + \delta_{B_X[0,r]})^*$ is differentiable at u^* with $\nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*) = u$, and so

$$\partial(f + \delta_{B_X[0,r]})^*(u^*) = \{\nabla(f + \delta_{B_X[\bar{x},r]})^*(u^*)\} = \{u\}.$$

Thus, (4.3) follows from (4.1) and (4.6), while (4.4) follows from (4.2). The proof is complete.

Remark. Under assumptions (4.1) and (4.2), one can prove that the mapping $u^* \mapsto \mathcal{A}(u^*) \cap V$ is single-valued on some neighborhood of 0 based on the idea of the proof of Kenderov's result [10, Proposition 2.6].

Applying Theorem 4.1 to $\mathcal{A}(u^*) = S(f_{u^*}, \bar{x}, r)$, we have the following corollary.

Corollary 4.1. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ and $p \in (0, +\infty)$. Then $\bar{x}$ is a tilt-stable p-order weak minimizer of f if and only if $\bar{x}$ is a tilt-stable p-order minimizer of f.*

Theorem 4.2. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ and $q \in (1, +\infty)$. Suppose that there exist $r, r', \delta, \kappa \in (0, +\infty)$ such that $\bar{x} \in S(f, \bar{x}, r)$ and*

$$\kappa d(x, S(f_{u^*}, \bar{x}, r))^q \leq f_{u^*}(x) - \inf_{u \in B_X[\bar{x},r]} f_{u^*}(u) \tag{4.8}$$

for all $x \in B_X(\bar{x}, r')$ and $u^* \in B_{X^*}(0, \delta)$, where f_{u^*} and $S(f_{u^*}, \bar{x}, r)$ are defined by (1.3) and (1.10), respectively. Then there exists $\delta' \in (0,\ \delta)$ such that

$$S(f_{u^*}, \bar{x}, r) \cap B_X\left(\bar{x}, \frac{r'}{4}\right) \subset S(f_{v^*}, \bar{x}, r) + \left(\frac{\lambda\|u^* - v^*\|}{\kappa}\right)^{\frac{1}{q-1}} B_X \tag{4.9}$$

for all $u^*, v^* \in B_{X^*}(0, \delta')$ and all $\lambda \in (1, +\infty)$.

Proof. By definition, we have, for any $u^* \in X^*$,

$$\begin{aligned} f_{u^*}(\bar{x}) - \inf_{u\in B_X[\bar{x},r]} f_{u^*}(u) &= f(\bar{x}) - \inf_{u\in B_X[\bar{x},r]} (f(u) - \langle u^*, u-\bar{x}\rangle) \\ &\le f(\bar{x}) - \inf_{u\in B_X[\bar{x},r]} (f(u) - \|u^*\| r) \\ &= \|u^*\| r \end{aligned}$$

as $f(\bar{x}) = \inf_{u\in B_X[\bar{x},r]} f(u)$ thanks to the assumption that $\bar{x} \in S(f,\bar{x},r)$. Hence there exists $\delta' \in (0,\ \delta)$ such that

$$f_{u^*}(\bar{x}) - \inf_{u\in B_X[\bar{x},r]} f_{u^*}(u) < \kappa\Big(\frac{r'}{4}\Big)^q \quad \forall u^* \in B_{X^*}(0,\delta').$$

It follows from (4.8) that $d(\bar{x}, S(f_{u^*},\bar{x},r)) < \frac{r'}{4}$, and so

$$d\left(z, S(f_{u^*},\bar{x},r)\cap B_X\left(\bar{x},\frac{r'}{4}\right)\right) < \frac{r'}{2} \quad \forall u^* \in B_{X^*}(0,\delta') \text{ and } \forall z \in B_X\left(\bar{x},\frac{r'}{4}\right). \tag{4.10}$$

Let $u^*, v^* \in B_{X^*}(0,\delta')$ and $x_{u^*} \in S(f_{u^*},\bar{x},r)\cap B_X\big(\bar{x},\frac{r'}{4}\big)$. Take a sequence $\{x^n_{v^*}\}$ in $S(f_{v^*},\bar{x},r)$ such that

$$d(x_{u^*}, S(f_{v^*},\bar{x},r)) = \lim_{n\to\infty} \|x_{u^*} - x^n_{v^*}\|. \tag{4.11}$$

This implies that

$$\begin{aligned} \limsup_{n\to\infty} \|x^n_{v^*} - \bar{x}\| &\le \lim_{n\to\infty} \|x^n_{v^*} - x_{u^*}\| + \|x_{u^*} - \bar{x}\| \\ &\le d\left(x_{u^*}, S(f_{v^*},\bar{x},r)\cap B_X\left(\bar{x},\frac{r'}{4}\right)\right) + \frac{r'}{4} < \frac{3r'}{4}, \end{aligned}$$

where the last inequality holds thanks to (4.10). Thus, without loss of generality, we assume that $x^n_{v^*} \in B_X(\bar{x},r')$ for all $n\in\mathbb{N}$. Noting that

$$f_{u^*}(x_{u^*}) = \inf_{u\in B_X[\bar{x},r]} f_{u^*}(u) \quad \text{and} \quad f_{v^*}(x^n_{v^*}) = \inf_{u\in B_X[\bar{x},r]} f_{v^*}(u) \quad \forall n\in\mathbb{N},$$

we have, by (4.8) (applied to x_{u^*}, f_{v^*} in place of x, f_{u^*}),

$$\kappa d(x_{u^*}, S(f_{v^*},\bar{x},r))^q \le f_{v^*}(x_{u^*}) - f_{v^*}(x^n_{v^*}) \quad \forall n\in\mathbb{N}$$

and, similarly

$$0 \le \kappa d(x^n_{v^*}, S(f_{u^*},\bar{x},r))^q \le f_{u^*}(x^n_{v^*}) - f_{u^*}(x_{u^*}) \quad \forall n\in\mathbb{N}.$$

Hence

$$\begin{aligned} \kappa d(x_{u^*}, S(f_{v^*},\bar{x},r))^q &\le f_{v^*}(x_{u^*}) - f_{v^*}(x^n_{v^*}) + f_{u^*}(x^n_{v^*}) - f_{u^*}(x_{u^*}) \\ &= \langle u^* - v^*, x_{u^*} - x^n_{v^*}\rangle, \end{aligned}$$

and it follows that

$$\kappa d(x_{u^*}, S(f_{v^*},\bar{x},r))^q \le \|u^* - v^*\|\, \|x_{u^*} - x^n_{v^*}\|.$$

This and (4.11) imply that

$$d(x_{u^*}, S(f_{v^*},\bar{x},r)) \le \left(\frac{\|u^*-v^*\|}{\kappa}\right)^{\frac{1}{q-1}}.$$

Since x_{u^*} is arbitrary in $S(f_{u^*},\bar{x},r)\cap B_X\big(\bar{x},\frac{r'}{4}\big)$, this shows that (4.9) holds for all $u^*, v^* \in B_{X^*}(0,\delta')$ and all $\lambda \in (1,+\infty)$. The proof is complete.

Theorem 4.3. *Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Let $\bar{x} \in \mathrm{dom}(f)$ be a minimizer of f and p be a positive number. Consider the following statements:*

(i) *$\bar{x}$ is a tilt-stable p-order weak minimizer of f.*
(ii) *$\bar{x}$ is a tilt-stable p-order minimizer of f.*
(iii) *$\bar{x}$ is a stable $\frac{1+p}{p}$-order weak sharp minimizer of f.*
(iv) *$\bar{x}$ is a stable $\frac{1+p}{p}$-order sharp minimizer of f.*
(v) *∂f is p-order metrically regular at $\bar{x}$ for 0.*
(vi) *∂f is p-order strongly metrically regular at $\bar{x}$ for 0.*
(vii) *f^* is $C^{1,p}$-smooth on a neighborhood of 0.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (vi) $\Rightarrow$ (v) *always hold. If, in addition, f is convex, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) *are true.*

Remark. If the convexity assumption is dropped, then each of the implications (iv) $\Rightarrow$ (vi) and (v) $\Rightarrow$ (vi) is not longer valid. Indeed, even in the isolated minimizer case, a counter-example of (iv) $\Rightarrow$ (vi) was already provided for the case when $p = 1$ (see [5]). In the case when the solution sets are isolated around $\bar{x}$, recall (by Theorem SH) that (v) always implies (iii); therefore, it is natural to ask, in the nonconvex and nonisolated case, whether or not (v) $\Rightarrow$ (iii) still holds. The answer to this question is negative, and an example of nonconvex f with a minimizer $\bar{x}$ is given at the end of this paper such that $\bar{x}$ is not a stable $\frac{1+p}{p}$-order weak sharp minimizer of f even though the subdifferential ∂f is p-order metrically regular at $\bar{x}$ for 0. This also shows that (v) does not necessarily imply (vi) (because it always holds that (vi) $\Rightarrow$ (iii)).

Proof of Theorem 4.3. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftarrow$ (vi) are immediate from Corollary 4.1 and Theorem SH, (iv) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (v) are trivial, while (iii) $\Rightarrow$ (ii) is immediate from Theorem 4.2 and Corollary 4.1. Next consider the case when f is convex. By the first assertion and Theorem SH, it suffices to show (v) $\Rightarrow$ (vii) *for the proof of the second assertion.* Now suppose that (v) holds, namely suppose that there exist $\delta, \delta', \tau \in (0, +\infty)$ such that

$$\tau d(x, (\partial f)^{-1}(x^*)) \le d(x^*, \partial f(x))^p \quad \forall (x, x^*) \in B_X(\bar{x}, \delta) \times B_{X^*}(0, \delta').$$

Then,

$$(\partial f)^{-1}(u^*) \cap B_X(\bar{x}, \delta) \subset (\partial f)^{-1}(x^*) + \frac{2}{\tau}\|x^* - u^*\|^p B_X \quad \forall u^*, x^* \in B_{X^*}(0, \delta'). \tag{4.12}$$

By the convexity of f, we have

$$(\partial f)^{-1}(u^*) \subset S(f_{u^*}, \bar{x}, r) \quad \forall (u^*, r) \in X^* \times (0, +\infty).$$

It follows from (4.12) and Theorem 4.1 that (vii) holds. The proof is complete.

We conclude with a counterexample to show that (v) does not imply (iii) in the general case. To do this, let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \dfrac{1}{2^{n+1}} - \left|x - \dfrac{3}{2^{n+1}}\right|, & \text{if } \dfrac{1}{2^n} < x \le \dfrac{1}{2^{n-1}} \ (n = 1, 2, \dots) \\ x - 1, & \text{if } 1 < x. \end{cases}$$

Then f is a nonnegative Lipschitz function with

$$f\left(\frac{1}{2^n}\right) = 0 \quad \text{and} \quad f\left(\frac{3}{2^{n+1}}\right) = \frac{1}{2^{n+1}} \quad \forall n \in \mathbb{N}.$$

Moreover, it is easy to verify that

$$\partial f(x)=\begin{cases}[-1,\ 1], & \text{if } x\in\{0\}\cup\left\{\frac{1}{2^{n-1}}:\ n\in\mathbb{N}\right\}\cup\left\{\frac{3}{2^{n+1}}:\ n\in\mathbb{N}\right\}\\ \{-1\}, & \text{if } x\in(-\infty,\ 0)\cup\bigcup_{n\in\mathbb{N}}\left[\frac{3}{2^{n+1}},\ \frac{1}{2^{n-1}}\right]\\ \{1\}, & \text{if } x\in\bigcup_{n\in\mathbb{N}}\left[\frac{1}{2^{n}},\ \frac{3}{2^{n+1}}\right]\cup(1,+\infty).\end{cases}$$

Hence

$$\frac{1}{2}\leq d(x^*,\partial f(x))\quad\forall x\in\left(-\frac{1}{2},\ \frac{1}{2}\right)\setminus\left(\{0\}\cup\left\{\frac{1}{2^{n-1}}:\ n\in\mathbb{N}\right\}\cup\left\{\frac{3}{2^{n+1}}:\ n\in\mathbb{N}\right\}\right)$$

and

$$(\partial f)^{-1}(x^*)=\{0\}\cup\left\{\frac{1}{2^{n-1}}:\ n\in\mathbb{N}\right\}\cup\left\{\frac{3}{2^{n+1}}:\ n\in\mathbb{N}\right\}\quad\forall x^*\in(-1,\ 1).$$

It follows that ∂f is p-order metrically regular at 0 for 0 (for any $p\in(0,+\infty)$). On the other hand, noting that 0 and each $\frac{1}{2^n}$ are global minimizers of $f, S(f,0,r)$ is a infinite set for any $r>0$. Hence, 0 is not a $\frac{1+p}{p}$-order stable sharp minimizer. This and Theorem 4.3 implies that 0 is not a $\frac{1+p}{p}$-order stable weak sharp minimizer.

References

[1] F.J. Aragón Artacho, M.H. Geoffroy, Characterization of metric regularity of subdifferentials, J. Convex Anal. 15 (2008) 365–380.
[2] J.F. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems, Springer-Verlag, New York, 2000.
[3] J.V. Burke, M.C. Ferris, Weak sharp minima in mathematical programming, SIAM J. Control Optim. 31 (1993) 1340–1359.
[4] A.L. Dontchev, R.T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis, Set-Valued Anal. 12 (2004) 79–109.
[5] D. Drusvyatskiy, A.S. Lewis, Tilt stability, uniform quadratic growth, and strong metric regularity of the subdifferential, SIAM J. Optim. 23 (2013) 256–267.
[6] D. Drusvyatskiy, B.S. Mordukhovich, T.T.A. Nghia, Second-order growth, tilt stability and metric regularity of the subdifferential, J. Convex Anal. 21 (2014) 1165–1192.
[7] M.C. Ferris, Weak sharp minima and penalty fucntions in mathematical programming (Ph.D. thesis), University of Cambridge, Cambridge, 1988.
[8] H. Frankowska, M. Quincampoix, Hölder metric regularity of set-valued maps, Math. Program. 132 (2012) 333–354.
[9] A.D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Surveys 55 (2000) 501–558.
[10] P. Kenderov, Semi-continuity of set-valued monotone mappings, Fund. Math. 88 (1975) 61–69.
[11] B. Kummer, Inclusions in general spaces: Hoelder stability, solution schemes and Ekeland's principle, J. Math. Anal. Appl. 358 (2009) 327–344.
[12] A.S. Lewis, S. Zhang, Partial smoothness, tilt stability, and generalized Hessians, SIAM J. Optim. 23 (2013) 74–94.
[13] G. Li, B.S. Mordukhovich, Hölder metric subregularity with applications to proximal point method, SIAM J. Optim. 22 (2012) 1655–1684.
[14] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation I, Springer-Verlag, Berlin, Heidelberg, 2006.
[15] B.S. Mordukhovich, T.T.A. Nghia, Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces, Nonlinear Anal. 86 (2013) 159–180.
[16] B.S. Mordukhovich, T.T.A. Nghia, Full Lipschitzian and Hölder stability in optimization with applications to mathematical programming and optimal control, SIAM J. Optim. 24 (2014) 1344–1381.
[17] B.S. Mordukhovich, T.T.A. Nghia, Second-order characterizations of tilt stability with applications to nonlinear optimization, Math. Program. 149 (2015) 83–104.
[18] B.S. Mordukhovich, J.V. Outrata, Tilt stability in nonlinear programming under Mangasarian-Fromovitz constraint qualification, Kybernetiika 49 (2013) 446–464.
[19] B.S. Mordukhovich, W. Ouyang, Higher-order metric subregularity and its applications, J. Global Optim. (2015) in press.
[20] B.S. Mordukhovich, R.T. Rockafellar, Second-order subdifferential calculus with application to tilt stability in optimization, SIAM J. Optim. 22 (2012) 95–986.
[21] R.A. Poliquin, R.T. Rockafellar, Tilt stability of a local minimum, SIAM J. Optim. 8 (1998) 287–299.
[22] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer-Verlag, Berlin, 1998.
[23] J.-C. Yao, X.Y. Zheng, Error bound and well-posedness with respect to an admissible function, Reprint.

[24] C. Zalinescu, Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces, in: Proc. 12th Baikal Internat. Conf. on Optimization Methods and their Appl., Irkutsk, Russia, 2001, pp. 272–284.
[25] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.
[26] X.Y. Zheng, K.F. Ng, Calmness for L-subsmooth multifunctions in Banach spaces, SIAM J. Optim. 19 (2009) 1648–1673.
[27] X.Y. Zheng, K.F. Ng, Metric subregularity and calmness for nonconvex generalized equations in Banach spaces, SIAM J. Optim. 20 (2010) 2119–2136.
[28] X.Y. Zheng, K.F. Ng, Hölder stable minimizers, tilt stability and Hölder metric regularity of subdifferentials, SIAM J. Optim. 25 (2015) 416–438.
[29] X.Y. Zheng, X.Q. Yang, Weak sharp minima for semi-infinite optimization problems with applications, SIAM J. Optim. 18 (2007) 573–588.