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QUANTITATIVE ANALYSIS FOR PERTURBED ABSTRACT INEQUALITY SYSTEMS IN BANACH SPACES*

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Abstract. Using the error bound results established in the present paper for approximate solutions, we study the stability issues when perturbed by possibly nonaffine smooth maps E for the abstract inequality system $F \geq_K 0$ defined by a (possible nonclosed) convex cone K and a Fréchet differentiable function F satisfying the (extended) weak γ-condition. We provide some sufficient conditions, in terms of the information at a solution x_0, for ensuring the lower semicontinuity and/or the Lipschitz-like continuity at x_0 of the solution mapping for the perturbed system $F + E \geq_K 0$ with smooth perturbation E. Explicit upper bounds of the Lipschitz-like moduli are also provided.

Key words. abstract inequality, error bound, lower semicontinuity, Lipschitz-like continuity

AMS subject classifications. 90C48, 90C31, 49K40, 90C26

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1. Introduction.

Given a suitable vector-valued mapping F between Banach spaces X and Y, and a convex "cone" K in Y (not necessarily closed and we allow the case in which $0 \notin K$) in the sense that K is convex and $\lambda z \in K$ for any $z \in K$ and $\lambda > 0$, we consider the abstract inequality

$$F(x) \geq_K 0, \tag{1.1}$$

and the associated perturbed one

$$F(x) + E(x) \geq_K 0, \tag{1.2}$$

where E is a suitable mapping, representing a perturbation, and not restricted to be an affine one. Let $\mathcal{S}(E)$ denote the solution set of (1.2) which, by definition, consists of all x satisfying (1.2), and let $S(= \mathcal{S}(0))$ denote the solution set of (1.1), namely that of (1.2) with zero E. Our interests in the present paper are mainly focused on how the presence of E would affect $\mathcal{S}(E)$, such as some possible properties of error bounds, and the lower semicontinuity and Lipschitz-like continuity of the map $\mathcal{S}(\cdot) : E \mapsto \mathcal{S}(E)$ (see section 4 for the definitions of these notions). Special cases of (1.1) include the following classical finite/infinite inequality system (1.3) and the inequality/strict-inequality system (1.4):

$$f_i(x) \geq 0, \qquad i \in I, \tag{1.3}$$

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(by taking $K = l_+(I) := \{(t_i) : t_i \geq 0\}$) and

$$f_i(x) > 0, \quad i \in I_P,$$
$$f_i(x) \geq 0, \quad i \in I_N, \tag{1.4}$$
$$f_i(x) = 0, \quad i \in I_E$$

(by taking $K := \{(t_i) : t_i > 0 \ \forall i \in I_P,\ t_i \geq 0 \ \forall i \in I_N, t_i = 0 \ \forall i \in I_E\}$), where $I := I_N \cup I_P \cup I_E$ is an index set, and the f_i's are suitable scalar-valued functions. The above two problems have many applications in several important problems, such as optimization, mathematical programming, and knowledge-based data classification; see, e.g., [13, 17, 25, 28, 40, 41, 47, 48] and references therein.

The notion of Lipschitz-like continuity was originally introduced by Aubin [1] under the name of the pseudo-Lipschitz property (though other names have been used such as the Aubin continuity property [15] and the sub-Lipschitzian property [51]) to quantify the stability of the solution set of a convex optimization problem. This notion, together with some other closely related notions including lower semicontinuity and calmness (two weaker properties than Lipschitz-like continuity) of the map $\mathcal{S}(\cdot)$, has been used in many optimization problems such as the semi-infinite/infinite linear inequality system, the linear complementarity problem, and constrained linear programming, among others (see [4, 5, 6, 7, 8, 9, 10, 12, 17, 18, 19, 23, 33]), and has also been studied extensively by using various variational tools such as metric regularity, generalized derivatives and coderivatives, etc., for more general (not necessarily affine) parametric optimization problems; see, for example, [2, 10, 16, 21, 23, 29, 30, 31, 32, 43, 53] and the references therein. The characterization issue of this notion and related ones has been studied mainly for the semi-infinite/infinite linear inequality system (1.3) with constant/affine perturbations, namely the perturbed system (1.2) with

$$Y := l^\infty(I), \quad K := l_+(I), \quad F \in \mathcal{A}(X, Y), \quad E \in Y \text{ or } E \in \mathcal{A}(X, Y),$$

where $\mathcal{A}(X, Y)$ is the space consisting of all continuous affine maps $A(\cdot) + y$ from X to Y endowed with the norm $\|A(\cdot) + y\| := \max\{\|y\|, \|A\|\}$ for each bounded linear operator A and $y \in Y$ (each $y \in Y$ may be viewed as an element of $\mathcal{A}(X, Y)$). Of particular relevance to our considerations, we mention the following aspects of the semi-infinite linear inequality system (1.3) for the map $\mathcal{S}(\cdot)$ on $X := \mathbb{R}^n$.

- *Lower semicontinuity.* Several characterizations for the lower semicontinuity of $\mathcal{S}(\cdot)$ were provided in [18, Theorem 3.1] and [5, Theorem 3.1] with affine perturbations, mainly in terms of the Slater condition, Robinson regularity, Robinson stability, and other well-known stability concepts in the literature.
- *Calmness.* A characterization for the calmness property of $\mathcal{S}(\cdot)$ with constant perturbations was derived in [6, Theorem 3] in terms of the ACQ and the uniform dual boundedness condition, and an operative expression for the calmness modulus was provided in [6, Theorem 2] for the case in which I is finite.
- *Lipschitz-like property.* Characterizations of the Lipschitz-like continuity of $\mathcal{S}(\cdot)$ with constant perturbations were studied in [17], while a formula for the Lipschitz-like modulus for $\mathcal{S}(\cdot)$ with affine perturbations was given in [11, Theorem 1].

Further extensions regarding the Lipschitz-like property are reported in [4, 26].

- *System* (1.3) *on Banach space X with constant perturbations.* Several characterizations for the Lipschitz-like property of $\mathcal{S}(\cdot)$ were given in [4, Theorem 4.1] in terms of the Slater/coderivative condition, and a precise formula for computing the exact bounds of the Lipschitz-like modulus was derived in [4, Theorem 4.6 and Corollary 4.7] in terms of the coderivative norm.

- *Abstract linear inequality system* (1.1) *on* $X = \mathbb{R}^n$ *with affine perturbations.* In the case in which K is closed, Huyen and Yen provided some equivalent conditions in [26, Theorem 3.9] for the local Lipschitz-like property of $\mathcal{S}(\cdot)$ in terms of the Robinson metric regularity property of $\mathcal{S}(\cdot)$, and as applications they also provided verifiable sufficient conditions for the Lipschitz-like property for the classical linear inequality system (1.3), the linear complementarity problem, and a class of affine variational inequalities under a linear perturbation by virtue of the coderivatives of implicit multifunctions.

We observe that all results mentioned above are obtained under the assumption that K is closed, and F and the perturbations E are affine. For classical problems such as (1.4), it is clear that the closeness assumption is too stringent. Our main purpose in the present paper is to deal with the more general case in which K is not necessarily closed, and F, E are given C^2-maps; we provide some sufficient conditions for ensuring the lower semicontinuity and/or the Lipschitz-like continuity of $\mathcal{S}(\cdot)$ for the perturbed system (1.2) at a solution x_0 of (1.1). These results are presented in section 4 and, to the best of our knowledge, almost all of them are new.

The approach used in the present paper is quite different from previous ones on this topic in the literature. To furnish the tools to establish our main results, we study first in section 3 the issues on the quantitative estimate of the error bound for an "approximate solution" x_0 of (1.1) in the sense that $F(x_0)+y_0 \geq_K 0$ for some $y_0 \in Y$ with "small" norm. The study of this issue is of independent interest and is in the spirit of the work of Dedieu [14] on the estimate of the error bound for an "approximate solution" of (1.1). The notion of the error bound for an "approximate solution" of (1.1) adopted by us here was introduced by Dedieu [14] and is not the same as (but is a modification of) the more famous notion of the error bound for (1.1) originating from Hoffman's error bound (see [20, 24]) for the inequality system (1.3) with each f_i being an affine function on the Euclidean space $\mathbb{R}^m$. Extensions of Hoffman's error bound results to the nonlinear inequality system (1.3) or to the abstract inequality system (1.1) when K is a closed convex cone have been well investigated in the literature; see, for example, [35, 38, 39, 48] and references therein. Recently, Li and Ng [35] extended the results of error bounds in [14] for an "approximate solution" of system (1.4) to the more general system (1.1) with K is not necessarily closed, which in particular improved the corresponding ones for system (1.4) in [14]. The results on the quantity estimate of the error bound for an "approximate solution" of (1.1) obtained in the present paper not only provide crucial tools for the study of the issue on the lower semicontinuity and the Lipschitz-like continuity of the solution mapping but also improve the main results in [35, Theorem 2.1] (and so improves further the corresponding ones in [14]); see Remark 3.2.

2. Preliminaries. We always assume that X, Y, Z are Banach spaces. Fix x in one of these spaces, and let D be a subset. We use $\mathrm{d}(x, D)$ to denote the distance from x to D. Let $\mathbb{B}(x, r)$ stand for the open ball with center x and radius r, and the corresponding closed ball is denoted by $\overline{\mathbb{B}}(x, r)$. In particular, we write $\mathbb{B} := \mathbb{B}(0, 1)$ and $\overline{\mathbb{B}} := \overline{\mathbb{B}}(0, 1)$. As usual, the space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. It is convenient to use the notation $\|D\|$ to denote its distance to the origin, that is,

$$\|D\| := \mathrm{d}(0, D) = \inf\{\|a\| : \ a \in D\}, \tag{2.1}$$

with the convention that $\|\emptyset\| = +\infty$. We also make the convention that $D + \emptyset = \emptyset$ for each set D. The projection on D is denoted by P_D, that is

$$P_D(x) := \{z \in D : \|x - z\| = \mathrm{d}(x, D)\} \quad \text{for any } x \in X.$$

The concept of a convex process (which was introduced by Rockafeller [49, 50] for convexity problems) plays a key role in the study of this paper.

Definition 2.1. *A set-valued map $T : X \to 2^Y$ is called a convex process from X to Y if it satisfies $0 \in T0$, $T(\lambda x) = \lambda Tx$, and $T(x+y) \supseteq Tx+Ty$ for all $x, y \in X$ and $\lambda > 0$.*

Thus $T : X \to 2^Y$ is a convex process if and only if its graph $\mathrm{Gr}(T) := \{(x, y) \in X \times Y : y \in Tx\}$ is a convex cone in $X \times Y$. By definition, a convex process $T : X \to 2^Y$ is closed if its graph $\mathrm{Gr}(T)$ is closed. As usual, the domain, range and inverse of a convex process T are respectively denoted by $\mathrm{D}(T)$, $\mathrm{R}(T)$, and T^{-1}, i.e.,

$$\mathrm{D}(T) := \{x \in X : Tx \neq \emptyset\}, \quad \mathrm{R}(T) := \bigcup\{Tx : x \in \mathrm{D}(T)\},$$

and

$$T^{-1}y := \{x \in X : y \in Tx\} \quad \text{for each } y \in Y.$$

Obviously T^{-1} is a convex process from Y to X. By definition, the following inequality holds for a convex process T:

$$\|T(x + y)\| \leq \|Tx\| + \|Ty\| \quad \text{for any } x, y \in X. \tag{2.2}$$

Definition 2.2. *Suppose that T is a convex process. The norm of T is defined by*

$$\|T\|_{\mathrm{d}} := \sup\{\|Tx\| : x \in \mathrm{D}(T), \ \ \|x\| \leq 1\}.$$

If $\|T\|_{\mathrm{d}} < +\infty$, we say that the convex process T is normed.

Let T, T_1, $T_2 : X \to 2^Y$ and $Q : Y \to 2^Z$ be convex processes. Recall that $T_1 \subseteq T_2$ means that $\mathrm{Gr}(T_1) \subseteq \mathrm{Gr}(T_2)$, that is, $T_1x \subseteq T_2x$ for each $x \in \mathrm{D}(T_1)$. By definition, one can verify easily that $\|T_1\|_{\mathrm{d}} \geq \|T_2\|_{\mathrm{d}}$ if $T_1 \subseteq T_2$ and $\mathrm{D}(T_1) = \mathrm{D}(T_2)$. Moreover, $T_1 \subseteq T_2$ if and only if $T_1^{-1} \subseteq T_2^{-1}$. The sum $T_1 + T_2$, multiple λT (with $\lambda \in \mathbb{R}$), and composite QT are processes defined respectively by

$$(T_1 + T_2)(x) := T_1x + T_2x, \quad (\lambda T)(x) := \lambda(Tx) \quad \text{for each } x \in X,$$

and

$$QT(x) = Q(T(x)) := \bigcup_{y \in T(x)} Q(y) \quad \text{for each } x \in X.$$

It is well known (and easy to verify) that $T_1 + T_2$, λT, and QT are also convex processes and the following assertions hold:

$$\|T_1+T_2\|_{\mathrm{d}} \leq \|T_1\|_{\mathrm{d}}+\|T_2\|_{\mathrm{d}}, \qquad \|QT\|_{\mathrm{d}} \leq \|Q\|_{\mathrm{d}}\,\|T\|_{\mathrm{d}}, \qquad \text{and} \qquad \|\lambda T\|_{\mathrm{d}} = |\lambda|\|T\|_{\mathrm{d}}. \tag{2.3}$$

For notational convenience, we introduce a larger norm $\|T\|$ for a convex process T:

$$\|T\| := \sup\{\|Tx\| : x \in \overline{\mathbb{B}}\}.$$

Clearly, one has the following implications:

$$[\|T\| < +\infty] \Rightarrow [\mathrm{D}(T) = X] \Rightarrow [\|T\|_{\mathrm{d}} = \|T\|] \ \text{ and } \ [\|QT\| < +\infty] \Rightarrow \mathrm{R}(T) \subseteq \mathrm{D}(Q). \tag{2.4}$$

The first assertion in the following proposition is a direct consequence of [46, Corollary, Page 131] and [45, Theorem 5] (see also [37, Proposition 2.2]), and the second one is known in [37, Proposition 2.1].

PROPOSITION 2.3. *Let $T : X \to 2^Y$ be a closed convex process. Then we have the following assertions:*

(i) *If $\mathrm{R}(T)$ is a closed linear subspace, then T^{-1} is normed.*
(ii) *If X is reflexive and T^{-1} is a closed normed convex process, then $\mathrm{R}(T)$ is closed.*

Letting Ω be a subset of X with nonempty interior denoted by $\mathrm{int}\Omega$ and $k \in \mathbb{N}$, the set of all natural numbers, we use $\mathcal{C}^k(\Omega, Y)$ to denote the set of all kth-order smooth operators, so $F \in \mathcal{C}^k(\Omega, Y)$ means that $F : \Omega \to Y$ and its kth Fréchet derivative is continuous. Associated with any pair (C, F) of a nonempty closed convex cone C in Y and $F \in \mathcal{C}^1(\Omega, Y)$, let $T_x : X \to 2^Y$ denote the process at $x \in \mathrm{int}\Omega$ defined by

$$T_x d := F'(x)d - C \quad \text{for each } d \in X, \tag{2.5}$$

and let its inverse process $T_x^{-1} : Y \to 2^X$ be defined by

$$T_x^{-1} y := \{d \in X :\ F'(x)d \in y + C\} \quad \text{for each } y \in Y. \tag{2.6}$$

Since $F'(x)$ is continuous and C is closed, it is easy to verify that T_x and T_x^{-1} are closed. In the following lemma, we list some useful properties for convex processes: (2.7) is a straightforward use of the fact that $C + C = C$; (2.8) holds by definition, while the proof for (2.9) is standard (see [36, Proposition 2.3], for example).

LEMMA 2.4. *Let $x_0 \in X$, $A \in \mathcal{L}(X, Y)$, and let T_{x_0} be defined by (2.5). Then one has that*

$$(T_{x_0} + A)^{-1} F'(x_0) T_{x_0}^{-1} \subseteq (T_{x_0} + A)^{-1}. \tag{2.7}$$

Furthermore, if $C = \overline{K}$ then, for any $-y \in \mathrm{R}(T_{x_0})$,

$$\|T_{x_0}^{-1}(-y)\| = \|T_{x_0}^{-1}(K - y)\| = \|T_{x_0}^{-1}(C - y)\| \le \|T_{x_0}^{-1}\| \mathrm{d}(y, C \cap (y + \mathrm{R}(T_{x_0}))), \tag{2.8}$$

where we adopt the convention that $+\infty \cdot 0 = 0$. Moreover, if it is additionally assumed that $\|T_{x_0}^{-1} A\| < 1$, then

$$\mathrm{R}(T_{x_0}) = \mathrm{R}(T_{x_0} + A) \quad \textit{and} \quad \|(T_{x_0} + A)^{-1} F'(x_0)\| \le \frac{1}{1 - \|T_{x_0}^{-1} A\|}. \tag{2.9}$$

For higher-order consideration, we introduce more notation. Let $k \in \mathbb{N}$ and consider a k-multilinear bounded operator $\Xi : (X)^k \to Y$. We define the norm $\|\Xi\|$ by

$$\|\Xi\| := \sup\{\|\Xi(x_1, \ldots, x_k)\| :\ (x_1, \ldots, x_k) \in (X)^k, \|x_i\| \le 1 \text{ for each } i\};$$

also, let $\mathrm{R}(\Xi)$ denote the image of Ξ:

$$\mathrm{R}(\Xi) := \{\Xi(x_1, \ldots, x_k) :\ (x_1, \ldots, x_k) \in (X)^k\}.$$

Assume that $F \in \mathcal{C}^k(\Omega, Y)$ and so the kth derivative $F^{(k)}(x)$ at $x \in \mathrm{int}\Omega$ is a k-multilinear bounded operator from $(X)^k$ to Y. It follows that, for any x_0, $x \in \mathrm{int}\Omega$, and for any $(z_1, z_2, \ldots, z_{k-1}) \in X^{k-1}$, $T_{x_0}^{-1}(F^{(k)}(x)(z_1, z_2, \ldots, z_{k-1}))$ is a convex process from X to Y. Define

$$\|T_{x_0}^{-1} F^{(k)}(x)\| := \sup\{\|T_{x_0}^{-1}(F^{(k)}(x)(z_1, z_2, \ldots, z_{k-1}))\| : \{z_i\}_{i=1}^{k-1} \subset \mathbb{B}\}. \tag{2.10}$$

Note in particular that, for each $j \le k$,

$$\|T_{x_0}^{-1} F^{(k)}(x) z^j\| \le \|T_{x_0}^{-1} F^{(k)}(x)\| \|z\|^j \quad \text{for each } z \in X, \tag{2.11}$$

where the z^j denotes, as usual, $(z, \ldots, z) \in (X)^j$ for each $z \in X$; moreover, if $(z_1, \ldots, z_l) \in (X)^l$, then $(z^j, z_1, \ldots, z_l)$ denotes the corresponding element in $(X)^{j+l}$. Thus, in terms of the notation $\mathrm{R}(\cdot)$, it is routine to verify that, for all x, $z \in \mathrm{int}\Omega$, the following equivalences and implication hold:

$$\begin{aligned} [\|T_z^{-1} F'(x)\| < +\infty] \Rightarrow [\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T_z)] &\Leftrightarrow [\mathrm{R}(T_x) \subseteq \mathrm{R}(T_z)] \\ &\Leftrightarrow [\mathrm{D}(T_z^{-1} F'(x)) = X]. \end{aligned} \tag{2.12}$$

In his study of problem (1.1) with $K = C$, a closed convex cone, Robinson [45] required an important assumption that T_{x_0} is surjective (henceforth to be referred to as the Robinson condition, as in [34]). We say that (cf. [36]) (T_{x_0}, F) satisfies the weak-Robinson condition on $\mathbb{B}(x_0, r)$ if $-F(x_0) \in \mathrm{R}(T_{x_0})$ and

$$\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T_{x_0}) \quad \text{for each } x \in \mathbb{B}(x_0, r). \tag{2.13}$$

For the case in which $F \in C^2(\Omega, Y)$ the notion of the γ-condition for F was first introduced by Wang [55] to study Smale's point estimate theory for operators which are not required to be analytic. This notion was also used in [22] to improve the corresponding results in [54]. An extended version of this notion given below will be useful in the presence of convex processes.

Definition 2.5. *Given*

$$x_0 \in X,\ \gamma \in [0, +\infty),\ r \in (0, +\infty) \text{ with } \gamma r \le 1, \tag{2.14}$$

let $T : X \to 2^Y$ *be a closed convex process. We say that* (T, F) *satisfies the weak* γ*-condition on* $\mathbb{B}(x_0, r)$ *if* $F \in C^2(\mathbb{B}(x_0, r), Y)$ *and*

$$\|T^{-1} F''(x)\| \le \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3} \quad \text{for each } x \in \mathbb{B}(x_0, r). \tag{2.15}$$

Note that (2.15) implies the inclusion

$$\mathrm{R}(F''(x)) \subseteq \mathrm{R}(T) \quad \text{for each } x \in \mathbb{B}(x_0, r). \tag{2.16}$$

The proof for Lemma 2.6 below is similar to that for [37, Proposition 3.1], and so is omitted here.

Lemma 2.6. *Given* (2.14)*, suppose that* X *is reflexive and* $\mathrm{R}(F'(x_0)) \subseteq \mathrm{R}(T)$*. Suppose that* (T, F) *satisfies the weak* γ*-condition on* $\mathbf{B}(x_0, r)$*. Then the following assertions hold.*

(i) (T, F) *satisfies* (2.13) *with* T *in place of* T_{x_0}*:*

$$\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T) \quad \text{for each } x \in \mathbb{B}(x_0, r). \tag{2.17}$$

(ii) *If also* $r \in (0, \frac{2-\sqrt{2}}{2\gamma}]$ *and* $x \in \mathbf{B}(x_0, r)$*, then*

$$T^{-1} \iint_{[0,1]^2} [\pm sF''(x_0 + ts(x - x_0))](x - x_0)^2 \,\mathrm{d}s\mathrm{d}t \neq \emptyset, \tag{2.18}$$

and

$$\|T^{-1}(F'(x) - F'(x_0))\| \leq -1 + \frac{1}{(1 - \gamma\|x - x_0\|)^2}. \tag{2.19}$$

3. Error bounds for approximate solutions. This section is devoted to the study on the quantitative estimate of the error bound for an "approximate solution" of (1.1). The main tool for our study in the present paper is [37, Theorem 1.1], which provides the convergence criterion and the error estimate for the sequence generated by the extended Newton method of solving the abstract inequality (3.6) below on reflexive Banach spaces defined by functions satisfying some basic assumptions (i.e., (3.1) and (3.2) below with $\Lambda = r^*$). Thus, for simplicity, we always assume, for the remainder of this paper, the following blanket assumption (collectively denoted by (3.1)):

$$\begin{cases} X \text{ is a reflexive Banach space}, \\ K \text{ is a convex "cone" (not necessarily closed) and } C := \overline{K}, \\ \gamma \in [0, +\infty) \text{ and } \Lambda \in (0, \infty) \text{ with } \gamma\Lambda \leq 1, \\ x_0 \in X \text{ and } F \in \mathcal{C}^2(\mathbb{B}(x_0, \Lambda), Y) \text{ such that } (T_{x_0}, F) \\ \qquad\qquad \text{satisfies the weak } \gamma\text{-condition on } \mathbb{B}(x_0, \Lambda), \\ \|T_{x_0}^{-1}\|_{\mathrm{d}} < \infty \text{ and } \xi := \|T_{x_0}^{-1}(-F(x_0))\| < +\infty \text{ (namely, } -F(x_0) \in \mathrm{R}(T_{x_0})). \end{cases} \tag{3.1}$$

Assuming

$$\gamma\xi \leq 3 - 2\sqrt{2}, \tag{3.2}$$

the majorizing function $h : [0, \frac{1}{\gamma}) \to \mathbb{R}$ defined by $h(t) := \xi - t + \frac{\gamma t^2}{1-\gamma t}$ for each $[0, \frac{1}{\gamma})$ has a nonpositive minimum value and so has two zeroes in $[0, \frac{1}{\gamma})$ (counting the multiplicity); see [37, 55]. Let r_0 and r^* denote the minimum point and the smaller zero of the function h, respectively, namely

$$r_0 := \frac{2 - \sqrt{2}}{2\gamma} \quad \text{and} \quad r^* := \frac{2\xi}{1 + \gamma\xi + \sqrt{(1 + \gamma\xi)^2 - 8\gamma\xi}}, \tag{3.3}$$

where, as usual, we adopt the convention that $\frac{a}{0} := +\infty$ for any $a > 0$. Note that

$$r^* \leq r_0, \quad r^* < \frac{1}{\gamma}, \quad \text{and} \quad r^* < r_0 \quad \text{if } \gamma\xi < 3 - 2\sqrt{2}. \tag{3.4}$$

Indeed, by (3.2) and (3.3), one has that

$$r^* \leq \frac{1 + \gamma\xi}{4\gamma} \leq \frac{1 + (3 - 2\sqrt{2})}{4\gamma} = \frac{2 - \sqrt{2}}{2\gamma} = r_0. \tag{3.5}$$

Since $\frac{2-\sqrt{2}}{2} < 1$, this implies that $r^* < \frac{1}{\gamma}$. Also the second inequality in (3.5) is strict if $\gamma\xi < 3 - 2\sqrt{2}$, and so (3.4) is shown.

Our main aim in this section is to consider the abstract inequality problem (1.1),

$$F(x) \geq_K 0,$$

together with the following one with $C = \overline{K}$:

$$F(x) \geq_C 0. \tag{3.6}$$

Let S and $S_{\rm cl}$ respectively denote the solution sets of (1.1) and (3.6), namely

$$S := \{x \in X : \; F(x) \in K\} \quad \text{and} \quad S_{\rm cl} := \{x \in X : \; F(x) \in \overline{K}\}.$$

The following theorem is known in [37, Theorem 1.1] and will be useful in our study.

THEOREM 3.1. *We assume that* (3.1) *and* (3.2) *hold, and suppose* $\Lambda = r^*$*, where* r^* *is defined as in* (3.3)*. Let* $\{x_k\}$ *be a sequence defined by*

$$x_{k+1} := x_k + d_k \quad \textit{for each } k = 0, 1, 2, \dots, \tag{3.7}$$

where each $d_k \in \mathfrak{D}(x_k) := \{d \in X : \; F(x_k) + F'(x_k)d \in C\}$ *is such that* $\|d_k\| = \|\mathfrak{D}(x_k)\|$*. Then* $\{x_k\}$ *is well defined and converges to a point* $x^* \in S_{\rm cl}$ *satisfying* $\|x_0 - x^*\| \leq r^*$*.*

For the following theorem and subsequent discussion, it will be convenient to let η denote the elementary function defined by

$$\eta(t) := \frac{2}{1 + t + \sqrt{(1+t)^2 - 8t}} \quad \text{for any } t \in [0, 3 - 2\sqrt{2}]. \tag{3.8}$$

By differential calculus, one can verify that it is a (strictly) increasing function on $[0, 3 - 2\sqrt{2}]$, and its inverse function $\eta^{-1} : [1, \frac{2+\sqrt{2}}{2}] \to [0, 3 - 2\sqrt{2}]$ is given by

$$\eta^{-1}(\tau) = \frac{\tau - 1}{\tau(2\tau - 1)} \quad \text{for any } \tau \in \left[1, \frac{2 + \sqrt{2}}{2}\right]. \tag{3.9}$$

THEOREM 3.2. *Suppose that there exists* τ *satisfying*

$$1 < \tau \leq \frac{2 + \sqrt{2}}{2} \quad \textit{and} \quad \gamma\xi \leq \frac{\tau - 1}{\tau(2\tau - 1)} \tag{3.10}$$

(so (3.2) *holds). We assume that* (3.1) *holds, and suppose* $\Lambda = r^*$*, where* r^* *is defined as in* (3.3)*. Then* $S_{\rm cl} \neq \emptyset$ *and*

$$\mathrm{d}(x_0, S_{\rm cl}) \leq \tau \|T_{x_0}^{-1}(-F(x_0))\| \leq \tau \|T_{x_0}^{-1}\|_{\rm d} \, \mathrm{d}(F(x_0), C \cap (F(x_0) + \mathrm{R}(T_{x_0}))). \tag{3.11}$$

Moreover, if (T_{x_0}, F) *additionally satisfies*

$$(F(x_0) + \mathrm{R}(T_{x_0})) \cap P_C(F(x_0)) \neq \emptyset, \tag{3.12}$$

then

$$\mathrm{d}(x_0, S_{\rm cl}) \leq \tau \|T_{x_0}^{-1}\|_{\rm d} \, \mathrm{d}(F(x_0), K) < +\infty. \tag{3.13}$$

Proof. In terms of the monotonic function η and its inverse η^{-1} defined in (3.8) and (3.9), one has from (3.10) that $\gamma\xi \le \eta^{-1}(\tau) \le \eta^{-1}(\frac{2+\sqrt{2}}{2}) = 3 - 2\sqrt{2}$, and, by definition of r^* given in (3.3), $r^* = \eta(\gamma\xi)\xi \le \tau\xi < +\infty$. Thus (3.2) holds, and one concludes by Theorem 3.1 that the sequence $\{x_n\}$ defined there converges to a solution $x^* \in S_{\rm cl}$ such that $\|x_0 - x^*\| \le r^* \le \tau\xi$, and so the first inequality in (3.11) is shown. Hence, thanks to the last assumption in (3.1) and by (2.8), we have that

$$\|T_{x_0}^{-1}(-F(x_0))\| \le \|T_{x_0}^{-1}\|_{\rm d}\, {\rm d}(F(x_0), C \cap (F(x_0) + {\rm R}(T_{x_0}))),$$

and so (3.11) is shown. Moreover, $(F(x_0) + {\rm R}(T_{x_0})) \cap C \supseteq (F(x_0) + {\rm R}(T_{x_0})) \cap P_C(F(x_0)) \neq \emptyset$ by (3.12). Therefore we have that

$$\begin{aligned} {\rm d}(F(x_0), C \cap (F(x_0) + {\rm R}(T_{x_0}))) &\le {\rm d}(F(x_0), (F(x_0) + {\rm R}(T_{x_0})) \cap P_C(F(x_0))) \\ &= {\rm d}(F(x_0), C) < +\infty. \end{aligned}$$

Making use of (3.11), we arrive at (3.13), and the proof is complete. □

Our next concern regards the corresponding result for the solutions of (1.1) rather than (3.6). Given a convex set D of X, recall that (cf. [56, Page 6]) the recession cone of D is the set ${\rm rec}\, D$ defined as

$${\rm rec}\, D := \{x \in X :\ x + D \subseteq D\};$$

a smaller one is the strong recession cone ${\rm srec}\, D$ of D which is defined as

$${\rm srec}\, D := \{x \in X :\ x + \overline{D} \subseteq D\}.$$

Clearly ${\rm srec}\, D \subset {\rm rec}\, D$ (and the equality holds when D is closed), and $0 \in {\rm srec} D$ if and only if D is closed (hence (3.14) below holds automatically in the case in which K is closed). Note also that ${\rm ri}\, D \subseteq {\rm srec}\, D \subseteq D$ if D is a convex cone, where ${\rm ri}\, D$ denotes the relative interior of D.

THEOREM 3.3. *Assume that* (3.1) *holds and suppose that* $\gamma\xi < 3 - 2\sqrt{2}$ *and* $\Lambda > r^*$. *Suppose that there exists* τ *satisfying* (3.10) *and suppose that*

$$\text{(3.14)} \qquad {\rm srec}\, K \cap {\rm R}(T_{x_0}) \neq \emptyset.$$

Then the following assertion holds for the solution set S *of* (1.1)*:* $S \neq \emptyset$ *and*

$$\text{(3.15)} \qquad {\rm d}(x_0, S) \le \tau\|T_{x_0}^{-1}(-F(x_0))\| \le \tau\|T_{x_0}^{-1}\|_{\rm d} {\rm d}(F(x_0), C \cap (F(x_0) + {\rm R}(T_{x_0}))).$$

Moreover, if (T_{x_0}, F) *is additionally assumed to satisfy* (3.12)*, then*

$$\text{(3.16)} \qquad {\rm d}(x_0, S) \le \tau\|T_{x_0}^{-1}\|_{\rm d}\, {\rm d}(F(x_0), K).$$

Proof. The second inequality in (3.15) follows from (2.8) (noting that $-F(x_0) \in {\rm R}(T_{x_0})$ by the last line of (3.1)). Below we show the first inequality. By (3.14), pick a sequence $\{c_n\}$ from the cone ${\rm srec}\, K \cap {\rm R}(T_{x_0})$ such that $c_n \to 0$; then $\|T_{x_0}^{-1}\|_{\rm d}\|c_n\| \to 0$. For each $n \in \mathbb{N}$, let $F_n : X \to Y$ be defined by $F_n(\cdot) := F(\cdot) - c_n$, and we consider problem (3.6) along with

$$\text{(3.17)} \qquad F_n(x) \ge_C 0.$$

Since $F_n' = F'$, the convex process T_{x_0} defined in (2.5) is unchanged when F is replaced by F_n, and we have that

(3.18)

$$\xi_n := \|T_{x_0}^{-1}(-F_n(x_0))\| \le \|T_{x_0}^{-1}(-F(x_0))\| + \|T_{x_0}^{-1}\|_{\rm d}\|c_n\| = \xi + \|T_{x_0}^{-1}\|_{\rm d}\|c_n\| \to \xi,$$

where the inequality holds because

$$T_{x_0}^{-1}(-F_n(x_0)) = T_{x_0}^{-1}(c_n - F(x_0)) \supseteq T_{x_0}^{-1}(-F(x_0)) + T_{x_0}^{-1}(c_n)$$

(noting that $c_n, -F(x_0) \in \mathrm{R}(T_{x_0})$). Let r_n^* be defined similarly to r^* but corresponding to (3.17) (i.e., defined in (3.3) with ξ_n in place of ξ), so

$$r_n^* = \frac{2\xi_n}{1+\gamma\xi_n+\sqrt{(1+\gamma\xi_n)^2-8\gamma\xi_n}} \to r^*. \tag{3.19}$$

Since $\Lambda > r^*$, there exists N_0 such that $\Lambda > r_n^*$ for all $n > N_0$. It follows from (3.1) that (T_{x_0}, F_n) satisfies the weak γ-condition on $\mathbb{B}(x_0, r_n^*)$ for all $n > N_0$. Below we consider two cases (Case I and Case II) separately.

Case I: $\gamma\xi < \frac{\tau-1}{\tau(2\tau-1)}$. In this case, we have from (3.18) that there exists $N > N_0$ such that $\gamma\xi_n < \frac{\tau-1}{\tau(2\tau-1)}$ for all $n > N$. For all such n, we apply Theorem 3.2 to F_n in place of F, and get that $\mathrm{d}(x_0, S_n) \le \tau\|T_{x_0}^{-1}(-F_n(x_0))\|$, where $S_n := \{x \in X : F_n(x) \in C\} \ne \emptyset$. Noting $S_n \subseteq S$ (because $F(x) \in c_n + C \subseteq K$ as $c_n \in \operatorname{srec} K$ and $C = \overline{K}$ whenever $x \in S_n$), it follows that $S \ne \emptyset$ and

$$\mathrm{d}(x_0, S) \le \tau\|T_{x_0}^{-1}(-F_n(x_0))\| = \tau\xi_n \to \tau\xi = \tau\|T_{x_0}^{-1}(-F(x_0))\|,$$

verifying the first inequality in (3.15) for Case I.

Case II: $\gamma\xi \not< \frac{\tau-1}{\tau(2\tau-1)}$ (i.e., $\gamma\xi = \frac{\tau-1}{\tau(2\tau-1)}$ by (3.10)). Since $\gamma\xi < 3-2\sqrt{2}$ and τ satisfies (3.10), we have that $\tau \in (1, \frac{2+\sqrt{2}}{2})$ (because $\frac{\tau-1}{\tau(2\tau-1)} = 3-2\sqrt{2}$ when $\tau = \frac{2+\sqrt{2}}{2}$). Consider $\varepsilon > 0$ sufficiently small such that $\tau_\varepsilon := \tau+\varepsilon < \frac{2+\sqrt{2}}{2}$. Noting $\gamma\xi < \frac{\tau_\varepsilon-1}{\tau_\varepsilon(2\tau_\varepsilon-1)}$ (by the monotonicity of the function η^{-1} in (3.9)), one applies the result established for Case I and gets that $\mathrm{d}(x_0, S) \le \tau_\varepsilon\|T_{x_0}^{-1}(-F(x_0))\|$. Letting $\varepsilon \to 0$, the first inequality in (3.15) is seen to hold for this case too.

Finally, if one assumes that (3.12) holds, then (3.16) follows directly from (3.15), as in the proof of Theorem 3.2. The proof is complete. ∎

Remark 3.1. Assume the following condition:

$$K - F(x_0) \subseteq \mathrm{R}(T_{x_0}). \tag{3.20}$$

Then it holds that

$$C - F(x_0) \subseteq \mathrm{R}(T_{x_0}) \tag{3.21}$$

(noting that $\mathrm{R}(T_{x_0})$ is closed by Proposition 2.3(ii) applied to T_{x_0}) and

$$\operatorname{srec} K \subseteq K \subseteq C \subseteq \mathrm{R}(T_{x_0}) \tag{3.22}$$

(so if $\operatorname{srec} K \ne \emptyset$ then (3.14) of Theorem 3.3 is satisfied), because when $c \in C$, one has, for any $n \in \mathbb{N}$, that $nc - F(x_0) \in \mathrm{R}(T_{x_0})$ and so $c - \frac{1}{n}F(x_0) \in \mathrm{R}(T_{x_0})$, and passing to the limit gives $c \in \mathrm{R}(T_{x_0})$; thus the last inclusion of (3.22) holds, while the first two inclusions are trivial.

COROLLARY 3.4. *Assume that* (3.1) *holds, and suppose that there exists* τ *satisfying* (3.10), $\Lambda \ge r^*$, *and that* (T_{x_0}, F) *satisfies* (3.20). *Then the solution set* S *of* (1.1) *is nonempty and satisfies* (3.16) *provided that either* K *is closed, or that* $\operatorname{srec} K \ne \emptyset$, $\Lambda > r^*$, *and* $\gamma\xi < 3-2\sqrt{2}$.

Proof. Since (T_{x_0}, F) satisfies (3.20), it follows from Remark 3.1 that (3.21) holds; this trivially implies that $\mathrm{d}(F(x_0), C\cap(F(x_0)+\mathrm{R}(T_{x_0}))) = \mathrm{d}(0, (C-F(x_0))\cap\mathrm{R}(T_{x_0})) = \mathrm{d}(F(x_0), K)$. Thus the conclusion follows from Theorems 3.2 and 3.3. ∎

Let X be a Hilbert space and $A : X \to Y$ be a bounded linear operator such that its image $\mathrm{R}(A)$ is complemented in Y (in the sense that there exists a bounded linear projection operator $Q : Y \to \mathrm{R}(A)$). Then, by [44], there exists a bounded linear operator, called a generalized inverse of A and denoted by A^+ (associated with Q), from Y into X such that

$$A\,A^+\,A = A, \quad A^+\,A\,A^+ = A^+, \quad A^+\,A = \mathbf{I} - \Pi_{\ker A}, \quad \text{and} \quad A\,A^+ = Q, \tag{3.23}$$

where Π_D denotes the orthogonal projection on subset D of X and $\mathbf{I}$ is the identify operator on X. In particular, in the case in which Y is also a Hilbert space and $\mathrm{R}(A)$ is closed, A^+ is just the unique Moore–Penrose generalized inverse $A^\dagger$. Recall that T_{x_0} is defined by (2.5) with $C := \overline{K}$ and K is a convex cone in Y.

Lemma 3.5. *Let X be a Hilbert space and $x_0 \in X$ be such that $\mathrm{R}(F'(x_0))$ is complemented in Y and*

$$K \cup \{-F(x_0)\} \subseteq \mathrm{R}(F'(x_0)). \tag{3.24}$$

Then the following assertions hold:

$$\mathrm{R}(T_{x_0}) = \mathrm{R}(F'(x_0)) - C = \mathrm{R}(F'(x_0)), \quad \|T_{x_0}^{-1}\|_{\mathrm{d}} \le \|F'(x_0)^+\| < +\infty, \tag{3.25}$$

and

$$\begin{aligned}\|T_{x_0}^{-1}(-F(x_0))\| = \|F'(x_0)^+(C - F(x_0))\| &= \|F'(x_0)^+(K - F(x_0))\| \\ &\le \|F'(x_0)^+\|\mathrm{d}(F(x_0), K).\end{aligned} \tag{3.26}$$

Proof. Since $C = \overline{K}$ and $\mathrm{R}(F'(x_0))$ is a closed linear subspace by assumption, it is easy to verify

$$C - F(x_0) \subseteq \mathrm{R}(F'(x_0)) \quad \text{and} \quad \|F'(x_0)^+(K - F(x_0))\| = \|F'(x_0)^+(C - F(x_0))\|. \tag{3.27}$$

Also, the inequality in (3.26) follows from definition. Moreover, noting that the graph of T_{x_0} contains that of $F'(x_0)$, one has that

$$F'(x_0)^+v \in T_{x_0}^{-1}v \quad \text{for any } v \in \mathrm{R}(F'(x_0)), \tag{3.28}$$

because if $v \in \mathrm{R}(F'(x_0))$ then $F'(x_0)(F'(x_0)^+v) = v$ by (3.23), and so we have $F'(x_0)(F'(x_0)^+v) \in v + C$, that is, $F'(x_0)^+v \in T_{x_0}^{-1}v$. Observe that (3.24) implies the equalities in (3.25) and that (3.28) implies the inequality in (3.25) and the inequality

$$\|T_{x_0}^{-1}(-F(x_0))\| \le \|F'(x_0)^+(C - F(x_0))\| \tag{3.29}$$

because

$$\|T_{x_0}^{-1}v\| \le \|F'(x_0)^+v\| \le \|F'(x_0)^+\|\|v\| \quad \text{for all } v \in \mathrm{D}(T_{x_0}^{-1}),$$

$T_{x_0}^{-1}(C-F(x_0)) = T_{x_0}^{-1}(-F(x_0))$ by (2.8), and $F'(x_0)^+(C-F(x_0)) \subseteq T_{x_0}^{-1}(C-F(x_0))$ by (3.27). To complete the proof, it remains to verify the first equality in (3.26), that is, the converse inequality of (3.29). To do this, let $u \in T_{x_0}^{-1}(-F(x_0))$. By definition, there exists $c \in C$ such that $F'(x_0)u = c - F(x_0)$. This implies that $\|F'(x_0)^+(c-F(x_0))\| \le \|u\|$ thanks to (3.23); consequently, $\|F'(x_0)^+(C-F(x_0))\| \le \|T_{x_0}^{-1}(-F(x_0))\|$, and the proof is complete. ∎

The following corollary improves and extends the main theorem [35, Theorem 2.1] ([35] needs the inclusion assumption $\mathrm{R}(F''(x)) \subseteq \mathrm{R}(F'(x_0))$ for all $x \in \mathbb{B}(x_0, \bar{r})$, and the Hilbert space assumption of Y, in addition to that of X; moreover, in the case in which K is not closed, [35] additionally needs the nonempty assumption $\mathrm{ri}K \neq \emptyset$ for the second conclusion there).

Corollary 3.6. *Let X, $x_0 \in X$, and F be as in Lemma 3.5. Assume further that (3.1) holds, and suppose that there exists τ satisfying (3.10) and $\Lambda \geq r^*$. Then the assertion*

$$S \neq \emptyset \quad \text{and} \quad \mathrm{d}(x_0, S) \leq \tau \|F'(x_0)^+\| \, \mathrm{d}(F(x_0), K) \tag{3.30}$$

holds, provided that either K is closed, or $\operatorname{srec} K \neq \emptyset$, $\Lambda > r^*$, and $\gamma\xi < 3 - 2\sqrt{2}$.

Proof. By (3.24), one checks from Lemma 3.5 that (3.20) holds and $\|T_{x_0}^{-1}\|_{\mathrm{d}} \leq \|F'(x_0)^+\| < +\infty$. Thus the conclusion follows from Corollary 3.4 and the proof is complete. ∎

Remark 3.2.

(a) Note that assumption (3.24) is stronger than assumption (3.20). In particular, in the case in which $\overline{\operatorname{span} K} = Y$, (3.24) implies that $F'(x_0)$ is of full range.

(b) Noting by the last line of (3.1) that $-F(x_0) \in \mathrm{R}(T_{x_0})$, one sees that (3.20) holds if $\mathrm{R}(T_{x_0})$ is a closed linear space. In particular, if $\mathrm{R}(T_{x_0})$ is closed and $\mathrm{R}(F'(x_0)) \cap \operatorname{qri} K \neq \emptyset$, then $\mathrm{R}(T_{x_0})$ is a closed linear space and so (3.20) holds, where $\operatorname{qri} K$ denotes the quasi relative interior of K that is defined as the set of all $\bar{x} \in K$ such that $\overline{\operatorname{cone}(K - \bar{x})}$ is a linear subspace (see [3] for this notion and related topics).

(c) The assumption $\operatorname{srec} K \neq \emptyset$ is strictly weaker than the assumption $\mathrm{ri}K \neq \emptyset$. For example, let $Y := l_2$ be the (infinite-dimensional) Hilbert sequence space, and $K := \{(x_i) \in l_2 : \ x_i > 0\}$ be the positive cone. Then $\operatorname{srec} K = K \neq \emptyset$ but $\mathrm{ri}K = \emptyset$.

(d) There are many examples for which Theorem 3.3 is applicable, but Corollary 3.6 and the results in [35] are not applicable (even in the case in which Y is a Hilbert space).

4. Perturbations and stability. Recall the blanket assumption in (3.1), and recall also that $\mathcal{S}(E)$ is the solution set of (1.2) for perturbation E. In this section, we will study the stability issue for the perturbed abstract inequality system (1.2) when the perturbation E is allowed from $\mathcal{C}^2(\mathbb{B}(x_0, \Lambda), Y)$, including the lower semicontinuity and the Lipschitz-like property of the map $\mathcal{S} : \mathcal{C}^2(\mathbb{B}(x_0, \Lambda), Y) \to X$. We begin with the following definition on the notions of lower semicontinuity and the Lipschitz-like property for set-valued mappings.

Definition 4.1. *Let $\Phi : Z \rightrightarrows X$ be a set-valued mapping between metric space Z and Banach space X and let $\bar{x} \in \Phi(\bar{z})$. We say that Φ is*

(a) *lower semicontinuous at $(\bar{z}, \bar{x})$ if for any neighborhood V of $\bar{x}$, there exists a neighborhood U of $\bar{z}$ such that $\Phi(z) \cap V \neq \emptyset$ for any $z \in U$;*

(b) *Lipschitz-like around $(\bar{z}, \bar{x})$ with modulus $\ell \geq 0$ if there are neighborhoods U of $\bar{z}$ and V of $\bar{x}$ such that*

$$\Phi(z_1) \cap V \subseteq \Phi(z_2) + \ell \|z_1 - z_2\| \overline{\mathbb{B}} \quad \text{for any } z_1, z_2 \in U. \tag{4.1}$$

We define the exact Lipschitzian bound of Φ around $(\bar{z}, \bar{x})$ by

$$\operatorname{lip}_Z \Phi(\bar{z}, \bar{x}) := \inf\{\ell \geq 0 : (4.1) \text{ holds for some } \{U, V\}\};$$

equivalently, in terms of the distance function,

$$\operatorname{lip}_Z\Phi(\bar z,\bar x)=\limsup_{(z,x)\to(\bar z,\bar y)}\frac{\mathrm{d}(x,\Phi(z))}{\mathrm{d}(z,\Phi^{-1}(x))},\tag{4.2}$$

where, as usual, we adopt the convention that $\frac{0}{0}:=0$. Thus we have that $\operatorname{lip}_Z\Phi(\bar z,\bar x)=+\infty$ if Φ is not Lipschitz-like around $(\bar z,\bar x)$; see [4, 27, 42, 52] for more details.

Let $r\in(0,\Lambda]$, and respectively define the "seminorms" $|||\cdot|||_r$ and $\|\cdot\|_r$ on $\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y)$ by

$$|||E|||_r:=\sup\{\|E(x)\|:\ x\in\mathbb{B}(x_0,r)\}\le+\infty$$

and

$$\|E\|_r:=\max\left\{\|E(x_0)\|,\ \|E'(x_0)\|,\ \sup_{x\in\mathbb{B}(x_0,r)}\|E''(x)\|\right\}\tag{4.3}$$

for any $E\in\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y)$. Note that

$$|||E|||_r\le(1+r+r^2)\|E\|_r\quad\text{for any } E\in\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y)\tag{4.4}$$

because, for each $x\in\mathbb{B}(x_0,r)$,

$$\|E(x)\|\le\|E(x_0)\|+\|E'(x_0)\|\|x-x_0\|+|||E''|||_r\|x-x_0\|^2\le(1+r+r^2)\|E\|_r.$$

The space $\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y)$ endowed with the seminorm $\|\cdot\|_r$ is denoted $\mathcal{C}^2_r(\mathbb{B}(x_0,\Lambda),Y)$. Moreover, introduce the subspace $\mathcal{H}^2(x_0;r)$ and the cone $\mathcal{H}^2_0(x_0;r)$ of $\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y)$ as follows:

$$\mathcal{H}^2(x_0;r):=\{E\in\mathcal{C}^2_r(\mathbb{B}(x_0,\Lambda),Y):\ \mathrm{R}(E'(x_0))\cup\mathrm{R}(E''(\mathbb{B}(x_0,r)))\subseteq\mathrm{R}(T_{x_0})\}$$

and

$$\mathcal{H}^2_0(x_0;r):=\{E\in\mathcal{H}^2(x_0;r):\ -E(x_0)\in\mathrm{R}(T_{x_0})\}.$$

4.1. Lower semicontinuity. For notational simplicity, we define

$$F_E(\cdot):=F(\cdot)+E(\cdot)\quad\text{for any } E\in\mathcal{C}^2(\mathbb{B}(x_0,\Lambda),Y).\tag{4.5}$$

Recall that $r_0=\frac{2-\sqrt{2}}{2\gamma}$, as defined by (3.3).

LEMMA 4.2. *Assume that* (3.1) *holds and* $\Lambda=r_0$. *Let* $\epsilon\in[0,+\infty)$ *and* $E\in\mathcal{C}^2(\mathbb{B}(x_0,r_0),Y)$ *be such that*

$$(T_{x_0},E)\ \textit{satisfies the weak } \epsilon\textit{-condition on } \mathbb{B}\left(x_0,\frac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right).\tag{4.6}$$

Then

$$(T_{x_0},F_E)\ \textit{satisfies the weak } (\gamma+\epsilon)\textit{-condition on } \mathbb{B}\left(x_0,\frac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right).\tag{4.7}$$

Moreover, if additionally it is assumed that $A\in\mathcal{L}(X,Y)$ *satisfies* $\|T_{x_0}^{-1}A\|<\frac{1}{2}$, *then*

$$(T_{x_0}+A,F_E)\ \textit{satisfies the weak } \tilde\gamma\textit{-condition on } \mathbb{B}(\tilde x_0,\tilde r_0)\ \textit{for any } \tilde x_0\in\mathbb{B}\left(x_0,\frac{3-2\sqrt{2}}{\gamma+\epsilon}\right),\tag{4.8}$$

where

$$\tilde{\gamma} := \frac{2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|\tilde{x}_0 - x_0\|)^3} \quad \textit{and} \quad \tilde{r}_0 := \frac{2-\sqrt{2}}{2\tilde{\gamma}}; \tag{4.9}$$

in particular,

$$(T_{x_0}+A, F_E) \textit{ satisfies the weak } 2(\gamma+\epsilon)\textit{-condition on } \mathbb{B}\left(x_0, \tfrac{2-\sqrt{2}}{4(\gamma+\epsilon)}\right). \tag{4.10}$$

Proof. Let $x \in \mathbb{B}(x_0, \frac{2-\sqrt{2}}{2(\gamma+\epsilon)})$. By the fourth line in assumption (3.1) and (4.6), we note by (2.16) that the domains of $T_{x_0}^{-1}F''(x)$ and $T_{x_0}^{-1}E''(x)$ are the whole of $X \times X$. This, together with the triangle inequality in (2.3) (noting (2.4)), implies that

$$\|T_{x_0}^{-1}F_E''(x)\| \leq \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3} + \frac{2\epsilon}{(1-\epsilon\|x-x_0\|)^3} \leq \frac{2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|x-x_0\|)^3}, \tag{4.11}$$

where F_E is defined by (4.5). Thus (4.7) is checked.

Now assume $A \in \mathcal{L}(X,Y)$ satisfies $\|T_{x_0}^{-1}A\| < \frac{1}{2}$. To show (4.8), let $\tilde{x}_0 \in \mathbb{B}(x_0, \frac{3-2\sqrt{2}}{\gamma+\epsilon})$ and note the following elementary equivalence:

$$\left[\frac{2-\sqrt{2}}{2}\cdot\frac{1-t}{2} \leq \frac{2-\sqrt{2}}{2} - t\right] \Longleftrightarrow [t \leq 3-2\sqrt{2}] \quad \text{for any } t \geq 0.$$

Applying this to $(\gamma+\epsilon)\|x_0-\tilde{x}_0\|$ in place of t (noting that $(\gamma+\epsilon)\|x_0-\tilde{x}_0\| < 3-2\sqrt{2}$), one gets that

$$\frac{2-\sqrt{2}}{2}\cdot\frac{1-(\gamma+\epsilon)\|x_0-\tilde{x}_0\|}{2(\gamma+\epsilon)} \leq \frac{2-\sqrt{2}}{2(\gamma+\epsilon)} - \|x_0-\tilde{x}_0\|.$$

Recalling the definitions of $\tilde{\gamma}$, $\tilde{r}_0$, and r_0 (see (4.9)), we have then that $\tilde{r}_0 \leq \frac{2-\sqrt{2}}{2(\gamma+\epsilon)} - \|x_0-\tilde{x}_0\|$. It follows that $\mathbb{B}(\tilde{x}_0,\tilde{r}_0) \subseteq \mathbb{B}(x_0, \frac{2-\sqrt{2}}{2(\gamma+\epsilon)})$. Let $x \in \mathbb{B}(\tilde{x}_0,\tilde{r}_0)$. Then $1-\tilde{\gamma}\|x-\tilde{x}_0\| > 0$ by (4.9), and so

$$\begin{aligned}
1-(\gamma+\epsilon)\|x-x_0\| &\geq 1-(\gamma+\epsilon)(\|\tilde{x}_0-x_0\|+\|x-\tilde{x}_0\|)\\
&= (1-(\gamma+\epsilon)\|\tilde{x}-x_0\|)(1-\frac{(\gamma+\epsilon)}{1-(\gamma+\epsilon)\|\tilde{x}-x_0\|}\|x-\tilde{x}_0\|)\\
&\geq (1-(\gamma+\epsilon)\|\tilde{x}-x_0\|)(1-\tilde{\gamma}\|x-\tilde{x}_0\|) > 0,
\end{aligned}$$

where we have used the fact that

$$\frac{(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|\tilde{x}-x_0\|)} \leq \frac{2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|\tilde{x}-x_0\|)^3} = \tilde{\gamma}.$$

Since $(T_{x_0}+A)^{-1}F'(x_0)T_{x_0}^{-1}F_E''(x) \subseteq (T_{x_0}+A)^{-1}F_E''(x)$ and $\|(T_{x_0}+A)^{-1}F'(x_0)\| < 2$ by (2.7) and (2.9) in Lemma 2.4, and making use of (4.7) and (4.9), together with the second inequality in (2.3), it follows that

$$\begin{aligned}
\|(T_{x_0}+A)^{-1}F_E''(x)\| &\leq \|(T_{x_0}+A)^{-1}F'(x_0)\|\|T_{x_0}^{-1}F_E''(x)\|\\
&\leq \frac{2\cdot 2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|x-x_0\|)^3}\\
&\leq \frac{2\cdot 2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|\tilde{x}-x_0\|)^3\cdot(1-\tilde{\gamma}\|x-\tilde{x}_0\|)^3}\\
&= \frac{2\tilde{\gamma}}{(1-\tilde{\gamma}\|x-\tilde{x}_0\|)^3}.
\end{aligned} \tag{4.12}$$

The proof is complete. ∎

THEOREM 4.3. *Assume that* (3.1), (3.14) *hold and* $\Lambda = r_0$. *Let* $\epsilon \in [0, +\infty)$ *and* $E \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ *be such that* (4.6) *holds and* $\|T_{x_0}^{-1} E'(x_0)\| < \frac{1}{2}$. *Suppose that there exists* $\tau \in (1, \frac{2+\sqrt{2}}{2}]$ *such that*

$$\|T_{x_0}^{-1}(-(F+E)(x_0))\| < \frac{\tau - 1}{4\tau(\gamma+\epsilon)(2\tau-1)}. \tag{4.13}$$

Then, the perturbed inequality (1.2) *is solvable, and its solution set* $\mathcal{S}(E)$ *satisfies*

$$\begin{aligned} \mathrm{d}(x_0, \mathcal{S}(E)) &\leq \frac{\tau\, \|T_{x_0}^{-1}(-(F+E)(x_0))\|}{1 - \|T_{x_0}^{-1} E'(x_0)\|} \\ &\leq \frac{\tau\, \|T_{x_0}^{-1}\|_{\mathrm{d}} \big[\mathrm{d}((F+E)(x_0), C \cap (F(x_0) + E(x_0) + \mathrm{R}(T_{x_0})))\big]}{1 - \|T_{x_0}^{-1} E'(x_0)\|}. \end{aligned} \tag{4.14}$$

If, further, $-E(x_0) \in \mathrm{R}(T_{x_0})$ *and* (T_{x_0}, F) *satisfies* (3.12), *then*

$$\mathrm{d}(x_0, \mathcal{S}(E)) \leq \frac{\tau \|T_{x_0}^{-1}\|_{\mathrm{d}}}{1 - \|T_{x_0}^{-1} E'(x_0)\|} (\mathrm{d}(F(x_0), K) + \|E(x_0)\|). \tag{4.15}$$

Proof. Let F_E be defined by (4.5), i.e., $F_E = F + E$. Then

$$(T_{x_0} + E'(x_0))(u) = F'(x_0)u - C + E'(x_0)(u) = F_E'(x_0) - C.$$

Thus $T_{x_0} + E'(x_0)$ is the convex process at x_0 as defined in (2.5) associated with the pair (C, F_E) in place of (C, F). Moreover, thanks to the assumption $\|T_{x_0}^{-1} E'(x_0)\| < \frac{1}{2}$, we have, by (2.9) of Lemma 2.4, that

$$\mathrm{R}(T_{x_0}) \subseteq \mathrm{R}(T_{x_0} + E'(x_0)), \quad \|(T_{x_0} + E'(x_0))^{-1} F'(x_0)\| \leq \frac{1}{1 - \|T_{x_0}^{-1} E'(x_0)\|} < 2, \tag{4.16}$$

and, by (4.10) of Lemma 4.2, that $(T_{x_0} + E'(x_0), F_E)$ satisfies the weak γ_E-condition on $\mathbb{B}(x_0, r_E)$, where $\gamma_E := 2(\gamma+\epsilon)$ and $r_E := (2-\sqrt{2})/2\gamma_E$ (that is, r_E is the number r_0 given by (3.3) with γ_E in place of γ). Let

$$\xi_E := \|(T_{x_0} + E'(x_0))^{-1}(-F_E(x_0))\|$$

and

$$r_E^* := \frac{2\xi_E}{1 + \gamma_E \xi_E + \sqrt{(1+\gamma_E\xi_E)^2 - 8\gamma_E\xi_E}}$$

(that is, ξ_E is the number ξ given in (3.1) but with F_E in place of F, and r_E^* is the number r^* given by (3.3) but with ξ_E, γ_E in place of ξ, γ). We next show that

$$\xi_E < \frac{\tau - 1}{\gamma_E \tau (2\tau - 1)} \quad \text{and} \quad r_E^* < r_E. \tag{4.17}$$

Indeed, applying (2.7) (to $T_{x_0} + E'(x_0)$ in place of T) and (4.16), one has that

$$\begin{aligned} \|(T_{x_0} + E'(x_0))^{-1}(-F_E(x_0))\| &\leq \|(T_{x_0} + E'(x_0))^{-1} F'(x_0)\| \|T_{x_0}^{-1}(-F_E(x_0))\| \\ &\leq \frac{\|T_{x_0}^{-1}(-F_E(x_0))\|}{1 - \|T_{x_0}^{-1} E'(x_0)\|} \\ &\leq 2\|T_{x_0}^{-1}(-F_E(x_0))\|; \end{aligned} \tag{4.18}$$

hence, thanks to (4.13),

$$\xi_E \le 2\|T_{x_0}^{-1}(-F_E(x_0))\| < \frac{\tau-1}{2\tau(\gamma+\epsilon)(2\tau-1)} = \frac{\tau-1}{\gamma_E\,\tau(2\tau-1)},$$

showing the first inequality in (4.17), and that $\gamma_E\xi_E < 3-2\sqrt{2}$ as $\frac{\tau-1}{\tau(2\tau-1)} \le 3-2\sqrt{2}$ (see (3.9)). It follows from (3.4) that $r_E^* < r_E \le r_0 = \Lambda$ and so (4.17) is shown. By Theorem 3.3 (applied to $F_E,\ \gamma_E,\ \xi_E,\ r_E$ in place of $F,\ \gamma,\ \xi,\ \Lambda$), we conclude that the perturbed inequality (1.2) is solvable with the solution set $\mathcal{S}(E)$ satisfying

$$\text{(4.19)} \qquad \mathrm{d}(x_0,\mathcal{S}(E)) \le \tau\|(T_{x_0}+E'(x_0))^{-1}(-F_E(x_0))\| \le \frac{\tau\,\|T_{x_0}^{-1}(-F_E(x_0))\|}{1-\|T_{x_0}^{-1}E'(x_0)\|},$$

where the last inequality is due to (4.18). Moreover, by inequality (4.13), we know that $-F_E(x_0)\in \mathrm{R}(T_{x_0})$, and so by (2.8) (applied $-F_E(x_0)$ in place of y),

$$\text{(4.20)} \qquad \|T_{x_0}^{-1}(-F_E(x_0))\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\big[\mathrm{d}(F_E(x_0), C\cap(F_E(x_0)+\mathrm{R}(T_{x_0})))\big].$$

Therefore (4.14) is shown by (4.19) and (4.20).

For (4.15), assume that $-E(x_0)\in\mathrm{R}(T_{x_0})$ and that (T_{x_0},F) satisfies (3.12). The former implies that $F(x_0)+\mathrm{R}(T_{x_0}) \subseteq F_E(x_0)+\mathrm{R}(T_{x_0})$, and so

$$\begin{aligned}\mathrm{d}(F_E(x_0), C\cap(F_E(x_0)+\mathrm{R}(T_{x_0}))) &\le \mathrm{d}(F_E(x_0), C\cap(F(x_0)+\mathrm{R}(T_{x_0})))\\ &\le \|E(x_0)\| + \mathrm{d}(F(x_0), C\cap(F(x_0)+\mathrm{R}(T_{x_0})))\\ &= \mathrm{d}(F(x_0),C)+\|E(x_0)\|,\end{aligned}$$

where the last equality is due to (3.12). This, together with (4.14), implies (4.15). The proof is complete. ∎

Corollary 4.4. *In addition to the assumptions made in Theorem* 4.3*, suppose further that* $x_0\in S_{cl}$ *and that* $-E(x_0)\in\mathrm{R}(T_{x_0})$. *Then*

$$\text{(4.21)} \qquad \overline{\mathbb{B}}\left(x_0, \frac{\tau\,\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\|}{1-\|T_{x_0}^{-1}E'(x_0)\|}\right)\cap\mathcal{S}(E)\ne\emptyset.$$

Proof. Thanks to the assumption $-E(x_0)\in\mathrm{R}(T_{x_0})$ and the assumption that (3.14) holds, we have that $\|T_{x_0}^{-1}(-E(x_0))\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\| < +\infty$. For (4.21), we assume, without loss of generality, that $x_0\notin\mathcal{S}(E)$, so $\|T_{x_0}^{-1}(-E(x_0))\|>0$. By the assumed (4.13), we take $\tau'\in(1,\tau)$ sufficiently near τ such that

$$\|T_{x_0}^{-1}(-(F+E)(x_0))\| < \frac{\tau'-1}{4\tau'(\gamma+\epsilon)(2\tau'-1)} < \frac{\tau-1}{4\tau(\gamma+\epsilon)(2\tau-1)}.$$

By (4.14) of Theorem 4.3 but applied to τ' in place of τ, one has

$$\mathrm{d}(x_0,\mathcal{S}(E)) \le \frac{\tau'\,\|T_{x_0}^{-1}(-(F+E)(x_0))\|}{1-\|T_{x_0}^{-1}E'(x_0)\|} \le \frac{\tau'\,\|T_{x_0}^{-1}(-E(x_0))\|}{1-\|T_{x_0}^{-1}E'(x_0)\|} < \frac{\tau\,\|T_{x_0}^{-1}(-E(x_0))\|}{1-\|T_{x_0}^{-1}E'(x_0)\|},$$

where the second inequality holds because $x_0\in S_{\mathrm{cl}}$. Hence, there exists $x^*\in\mathcal{S}(E)$ such that

$$\|x^*-x_0\| < \frac{\tau\,\|T_{x_0}^{-1}(-E(x_0))\|}{1-\|T_{x_0}^{-1}E'(x_0)\|} \le \frac{\tau\,\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\|}{1-\|T_{x_0}^{-1}E'(x_0)\|}.$$

Therefore, (4.21) is shown. ∎

For Corollary 4.6 below, we need to recall the following lemma, which is known in [57, Theorem 3.1] for the Hilbert space setting but the proof presented there can be modified to suit the general Banach space setting.

LEMMA 4.5. *Let D and Z be two closed convex subsets of Y with $0 \in D \cap Z$ and let $\rho > 0$ be such that $\rho\mathbb{B} \subseteq Z$. Then, for any $r > 0$, one has that*

$$\mathrm{d}(y, D \cap Z) \leq \left(1 + \frac{4r}{\rho}\right) \max\{\mathrm{d}(y, D), \mathrm{d}(y, Z)\} \quad \textit{for any } y \in r\mathbb{B}. \tag{4.22}$$

The following corollary is a consequence of Theorem 4.3 and will be useful.

COROLLARY 4.6. *Assume that* (3.1), (3.14) *hold and $\Lambda = r_0$. Let $\epsilon \in (0, +\infty)$, $\tau = (2 + \sqrt{2})/2$, and suppose further that $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$. Then there exist $\delta \in (0, +\infty)$ and $\mu \in [1, +\infty)$ such that*

$$\|T_{x_0}^{-1}(-(F + E)(x_0))\| \leq \mu\|T_{x_0}^{-1}\|_{\mathrm{d}}(\mathrm{d}(F(x_0), K) + \|E(x_0)\|) \tag{4.23}$$

and

$$\mathrm{d}(x_0, \mathcal{S}(E)) \leq \frac{\tau\mu\|T_{x_0}^{-1}\|_{\mathrm{d}}}{1 - \|T_{x_0}^{-1}E'(x_0)\|}(\mathrm{d}(F(x_0), K) + \|E(x_0)\|) \tag{4.24}$$

whenever $E \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ with $\|E(x_0)\| < \delta$ satisfies (4.6), (4.13) *and that $\|T_{x_0}^{-1}E'(x_0)\| < \frac{1}{2}$.*

Proof. By the assumption $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$, we take $\delta > 0$ such that $2\delta\mathbb{B} \subseteq F(x_0) + \mathrm{R}(T_{x_0})$, and set $\mu := 1 + 4(\|F(x_0)\| + \delta)/\delta$. Below we show that δ and μ are as desired. Indeed, let $E \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ with $\|E(x_0)\| < \delta$ satisfy (4.6), (4.13), and that $\|T_{x_0}^{-1}E'(x_0)\| < \frac{1}{2}$, and let F_E be defined by (4.5). Then $\|F_E(x_0)\| \leq \|F(x_0)\| + \delta$ and $\mathbb{B}(-F_E(x_0), \delta) \subseteq \mathbb{B}(-F(x_0), 2\delta) \subseteq \mathrm{R}(T_{x_0})$; hence $\delta\mathbb{B} \subseteq F_E(x_0) + \mathrm{R}(T_{x_0})$. Applying Lemma 4.5 to C, $F_E(x_0) + \mathrm{R}(T_{x_0})$ in place of D, Z, we conclude that

$$\mathrm{d}(F_E(x_0), C \cap (F_E(x_0) + \mathrm{R}(T_{x_0}))) \leq \mu\mathrm{d}(F_E(x_0), C) \leq \mu(\mathrm{d}(F(x_0), K) + \|E(x_0)\|),$$

and it follows from (2.8) (applied to $F_E(x_0)$ in place of y) that (4.23) holds. Hence (4.24) holds by Theorem 4.3, since E is assumed to satisfy (4.6) and (4.13). The proof is complete. ∎

THEOREM 4.7. *Assume that* (3.1), (3.14) *hold and $\Lambda = r_0$. Suppose that $x_0 \in S$ and let $R \in (0, r_0]$. Then*

(i) *the solution map $\mathcal{S}(\cdot) : \mathcal{H}_0^2(x_0; R) \rightrightarrows X$ is lower semicontinuous at $(0, x_0)$,*
(ii) *the solution map $\mathcal{S}(\cdot) : \mathcal{H}^2(x_0; R) \rightrightarrows X$ is lower semicontinuous at $(0, x_0)$ provided that $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$.*

Proof. Let $\epsilon \in (0, 1)$ be arbitrary. Let $\delta := 1$ and $\mu := 1$ if $-F(x_0) \notin \mathrm{int}(\mathrm{R}(T_{x_0}))$, and otherwise, let $\delta \in (0, 1)$ and $\mu \in [1, +\infty)$ be the numbers determined by Corollary 4.6: if $E \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ with $\|E(x_0)\| < \delta$ satisfies (4.6), (4.13) (with $\tau := \frac{2+\sqrt{2}}{2}$) and that $\|T_{x_0}^{-1}E'(x_0)\| < \frac{1}{2}$, then (4.23) and (4.24) hold. Now choose $\delta_\epsilon \in (0, \delta)$ such that $\|T_{x_0}^{-1}\|_{\mathrm{d}}\, \delta_\epsilon < \min\{\frac{\epsilon}{4\mu}, \frac{3-2\sqrt{2}}{4\mu(\gamma+\epsilon)}\}$. By the definitions of $\|E\|_R$ (as in (4.3)) and δ_ϵ (noting that $\mu \geq 1$), one has that if

$$E \in \mathcal{H}^2(x_0; R) \quad \text{with} \quad \|E\|_R < \delta_\epsilon, \tag{4.25}$$

then

$$\|E(x_0)\| < \delta, \quad \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\delta_\epsilon < \min\left\{\frac{\epsilon}{4\mu}, \frac{3-2\sqrt{2}}{4\mu(\gamma+\epsilon)}\right\}, \tag{4.26}$$

$$\|T_{x_0}^{-1}E'(x_0)\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E'(x_0)\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\delta_\epsilon < \frac{\epsilon}{4\mu} < \frac{1}{2}, \tag{4.27}$$

and, if $R = r_0 = \frac{2-\sqrt{2}}{2\gamma}$ (and so $R > \frac{2-\sqrt{2}}{2(\gamma+\epsilon)}$),

$$\|T_{x_0}^{-1}E''(x)\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E''(x)\| < \frac{\epsilon}{4\mu} < \frac{2\epsilon}{(1-\epsilon\|x-x_0\|)^3} \quad \text{for each } x \in \mathbb{B}\left(x_0, \frac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right). \tag{4.28}$$

For the rest of our proof, we suppose that (4.25) holds. To complete the proof, we will verify that

$$\mathbb{B}(x_0,\epsilon) \cap \mathcal{S}(E) \ne \emptyset \tag{4.29}$$

for each of the following cases: (i) $E \in \mathcal{H}_0^2(x_0; R)$ and (ii) $-F(x_0) \in \operatorname{int}(\mathrm{R}(T_{x_0}))$. Recalling the assumption that $R \in (0, r_0]$, let us first consider the special case in which $R = r_0$.

For case (i), the present E satisfies $-E(x_0) \in \mathrm{R}(T_{x_0})$, and observe that, with $\tau := (2+\sqrt{2})/2$ (so $\frac{\tau-1}{\tau(2\tau-1)} = 3-2\sqrt{2}$), all the assumptions in Theorem 4.3 are met. Indeed, (4.6) holds by (4.28), and $\|T_{x_0}^{-1}E'(x_0)\| < \frac{1}{2}$ by (4.27). To check (4.13), note first that, by the assumption $x_0 \in S$, one has $0 \in -F(x_0) + K$ and so, by (2.6), $0 \in T_{x_0}^{-1}(-F(x_0))$ and $\|T_{x_0}^{-1}(-F(x_0))\| = 0$. This implies that

$$\|T_{x_0}^{-1}(-(F+E)(x_0))\| \le \|T_{x_0}^{-1}(-F(x_0))\| + \|T_{x_0}^{-1}(-E(x_0))\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\|. \tag{4.30}$$

Thus, noting that $\mu \ge 1$, one has by (4.26) that

$$\|T_{x_0}^{-1}(-(F+E)(x_0))\| \le \mu\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\| < \frac{\tau-1}{4(\gamma+\epsilon)\tau(2\tau-1)}. \tag{4.31}$$

Therefore we can apply (4.14) of Theorem 4.3 to conclude that $\mathcal{S}(E) \ne \emptyset$ and

$$\mathrm{d}(x_0, \mathcal{S}(E)) \le \frac{\tau\mu\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\|}{1-\|T_{x_0}^{-1}E'(x_0)\|} \le 4\mu\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E(x_0)\| < \epsilon \tag{4.32}$$

thanks to (4.26), (4.27) and noting that $\tau \le 2$. Thus (4.29) is proved for case (i). For case (ii), the proof is much the same but we use Corollary 4.6 in place of Theorem 4.3. In particular, noting $\mathrm{d}(F(x_0), K) = 0$ as $F(x_0) \in K$, we see that (4.31) and (4.32) continue to hold by (4.23) and (4.24), respectively. Therefore, we have completed the proof for the case in which $R = r_0$.

It remains to prove the same for the case in which $R \ne r_0$ (and so $R < r_0$ as $R \in (0, r_0]$ by assumption). In this case, set

$$\gamma' := \frac{2-\sqrt{2}}{2R} \quad \text{and} \quad r_0' := \frac{2-\sqrt{2}}{2\gamma'}.$$

Then $\gamma' > \gamma$, and so $r_0' \le r_0$ and $r_0' = R < \frac{1}{\gamma'}$. Hence (T_{x_0}, F) satisfies the weak γ'-condition on $\mathbb{B}(x_0, r_0')$. Thus one applies the result just established (with γ' in place of γ), and we conclude that (4.29) also holds for the case in which $R \ne r_0$. The proof is complete. $\square$

Recalling that (3.14) holds automatically in the case in which K is closed, we have the following corollary.

COROLLARY 4.8. *Assume* (3.1) *and* $\Lambda = r_0$. *Suppose that* $x_0 \in S$, *and let* $R \in (0, r_0]$. *If* K *is closed, then conclusions* (i) *and* (ii) *in Theorem* 4.7 *hold.*

4.2. Lipschitz-like continuity. To prepare the main theorem in this subsection, it is convenient to note the following simple lemma.

LEMMA 4.9. *Let* $\bar{\gamma} \in [0, +\infty)$, *and* $F_1, F_2 \in \mathcal{C}^2(\mathbb{B}(x_0, \frac{2-\sqrt{2}}{2\bar{\gamma}}), Y)$ *be such that each* (T_{x_0}, F_i) *satisfies the weak* $\bar{\gamma}$*-condition on* $\mathbb{B}(x_0, \frac{2-\sqrt{2}}{2\bar{\gamma}})$ *and that*

$$\mathrm{R}(F_i'(x_0)) \subseteq \mathrm{R}(T_{x_0}) \quad \textit{for each } i = 1, 2. \tag{4.33}$$

Then, for any $\tilde{x}_0 \in \mathbb{B}(x_0, \frac{2-\sqrt{2}}{2\bar{\gamma}})$, *the following equivalence holds:*

$$F_1(\tilde{x}_0) - F_2(\tilde{x}_0) \in \mathrm{R}(T_{x_0}) \Longleftrightarrow F_1(x_0) - F_2(x_0) \in \mathrm{R}(T_{x_0}). \tag{4.34}$$

Proof. Fix $\tilde{x}_0 \in \mathbb{B}(x_0, \frac{2-\sqrt{2}}{2\bar{\gamma}})$ and consider $i = 1, 2$. Then, by assumption, and using (2.18) of Lemma 2.6 (applied to $\frac{2-\sqrt{2}}{2\bar{\gamma}}$, $\tilde{x}_0$, $\bar{\gamma}$ in place of r, x, γ), we hence get that

$$\pm \iint_{[0,1]^2} [sF_i''(x_0 + ts(\tilde{x}_0 - x_0))](\tilde{x}_0 - x_0)^2 \,\mathrm{d}s\mathrm{d}t \in \mathrm{R}(T_{x_0}). \tag{4.35}$$

By Taylor's formula, we also have that

$$F_i(\tilde{x}_0) = F_i(x_0) + F_i'(x_0)(\tilde{x}_0 - x_0) + \iint_{[0,1]^2} [sF_i''(x_0 + ts(\tilde{x}_0 - x_0))](\tilde{x}_0 - x_0)^2 \,\mathrm{d}s\mathrm{d}t,$$

and so

$$\begin{aligned} F_1(\tilde{x}_0) - F_2(\tilde{x}_0) = {} & F_1(x_0) - F_2(x_0) + (F_1'(x_0) - F_2'(x_0))(\tilde{x}_0 - x_0) \\ & + \iint_{[0,1]^2} [sF_1''(x_0 + ts(\tilde{x}_0 - x_0))](\tilde{x}_0 - x_0)^2 \,\mathrm{d}s\mathrm{d}t \\ & - \iint_{[0,1]^2} [sF_2''(x_0 + ts(\tilde{x}_0 - x_0))](\tilde{x}_0 - x_0)^2 \,\mathrm{d}s\mathrm{d}t, \end{aligned}$$

which, together with (4.33) and (4.35), implies (4.34), as $\mathrm{R}(T_{x_0})$ is a cone. The proof is complete. $\square$

THEOREM 4.10. *Assume that* (3.1), (3.14) *hold and* $\Lambda = r_0 := \frac{2-\sqrt{2}}{2\gamma}$. *Let* $\epsilon \in [0, +\infty)$, $r \in (0, \frac{2-\sqrt{3}}{2(\gamma+\epsilon)})$, *and* $x_0 \in S_{\mathrm{cl}}$ *(that is* $F(x_0) \in \overline{K} = C$*).*

(i) *Suppose that* $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$. *Then there exists a pair* (δ, μ) *of positive numbers with* $\delta \le r$ *such that*

$$\mathbb{B}(x_0, \delta) \cap \mathcal{S}(E_1) \subseteq \mathcal{S}(E_2) + \frac{\mu \, \|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r}{2 - \|T_{x_0}^{-1} E_2'(x_0)\| - 1/(1 - (\gamma + \epsilon) r)^2} \overline{\mathbb{B}}, \tag{4.36}$$

whenever $E_1, E_2 \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ *with* $|||E_1|||_r$, $|||E_2|||_r < \delta$ *have the following properties:*
(4.37)

$$\|T_{x_0}^{-1} E_i'(x_0)\| < \frac{1}{6}, \quad (T_{x_0}, E_i) \text{ satisfies the weak } \epsilon\text{-condition on } \mathbb{B}\left(x_0, \tfrac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right).$$

(ii) *Let* $\tau \in (1, \frac{2+\sqrt{2}}{2}]$. *Then* (4.36) *holds with* $\delta = r$ *and* $\mu = \tau$ *whenever* $E_1, E_2 \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ *are such that* $E_1(x_0) - E_2(x_0) \in \mathrm{R}(T_{x_0})$ *and have the properties* (4.37) *and the following* (4.38)*:*

$$\|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_i|||_r < \frac{\tau - 1}{8\sqrt{3}(\gamma+\epsilon)\tau(2\tau-1)}. \tag{4.38}$$

Proof. Let $E_1, E_2 \in \mathcal{C}^2(\mathbb{B}(x_0, r_0), Y)$ satisfy (4.37), and $F_{E_i} := F + E_i$ be given by (4.5) with E_i in place of E for $i = 1, 2$. By the assumed γ-condition in (3.1) and the ϵ-conditions in (4.37), we see from (4.7) of Lemma 4.2 (applied to E_i in place of E) that
(4.39)

$$(T_{x_0}, F_{E_i}) \text{ satisfies the weak } (\gamma+\epsilon)\text{-condition on } \mathbb{B}\left(x_0, \tfrac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right) \text{ for each } i = 1, 2.$$

Further, (4.37) implies in particular that $\mathrm{R}(E_i'(x_0)) \subseteq \mathrm{R}(T_{x_0})$, and so $\mathrm{R}(F_{E_i}'(x_0)) \subseteq \mathrm{R}(T_{x_0})$ (noting by definition that $\mathrm{R}(F'(x_0)) \subseteq \mathrm{R}(T_{x_0})$). It follows from (2.19) of Lemma 2.6 (applied to (T_{x_0}, F_{E_i}) and $\frac{2-\sqrt{2}}{2(\gamma+\epsilon)}$ in place of (T_{x_0}, F) and r) that

$$\|T_{x_0}^{-1}(F_{E_i}'(x) - F_{E_i}'(x_0))\| \le -1 + \frac{1}{(1-(\gamma+\epsilon)\|x - x_0\|)^2} \quad \text{for each } x \in \mathbb{B}\left(x_0, \frac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right). \tag{4.40}$$

To prove (4.36) in (i) and (ii) we consider an arbitrary

$$\tilde{x}_0 \in \mathbb{B}(x_0, r) \cap \mathcal{S}(E_1), \tag{4.41}$$

and correspondingly set

$$A := E_2'(\tilde{x}_0) + F'(\tilde{x}_0) - F'(x_0), \quad \tilde{\xi} := \|(T_{x_0} + A)^{-1}(-F_{E_2}(\tilde{x}_0))\|, \tag{4.42}$$

and $\tilde{\gamma}, \tilde{r}_0$ as in (4.9). Then $\gamma \le \tilde{\gamma}$ and $\tilde{r}_0 \le r_0$. It is routine to check that $F_{E_2}' = F' + E_2'$ and that $(T_{x_0} + A)(\cdot) = F_{E_2}'(\tilde{x}_0)(\cdot) - C$ and so $T_{x_0} + A$ is the convex process at $\tilde{x}_0$ as defined in (2.5) associated with the pair (C, F_{E_2}) in place of (C, F). Since $r \le \frac{2-\sqrt{3}}{2(\gamma+\epsilon)} \le \frac{3-2\sqrt{2}}{\gamma+\epsilon} \le \frac{2-\sqrt{2}}{2(\gamma+\epsilon)}$, we have by (4.41) that

$$\tilde{x}_0 \in \mathbb{B}(x_0, r) \subseteq \mathbb{B}\left(x_0, \frac{3-2\sqrt{2}}{\gamma+\epsilon}\right), \tag{4.43}$$

and it follows from (4.40) that

$$\|T_{x_0}^{-1}(F_{E_2}'(\tilde{x}_0) - F_{E_2}'(x_0))\| \le -1 + \frac{1}{(1-(\gamma+\epsilon)\|\tilde{x}_0 - x_0\|)^2} < -1 + \frac{1}{(1-\frac{2-\sqrt{3}}{2})^2} = \frac{1}{3}.$$

Since $T_{x_0}^{-1}$ is convex process and noting the definition of A in (4.42), it follows from (4.37) and the triangle inequality that

$$\begin{aligned}\|T_{x_0}^{-1}A\| &\le \|T_{x_0}^{-1}E_2'(x_0)\| + \|T_{x_0}^{-1}(F_{E_2}'(\tilde{x}_0) - F_{E_2}'(x_0))\| \\ &< \|T_{x_0}^{-1}E_2'(x_0)\| - 1 + \frac{1}{(1-(\gamma+\epsilon)r)^2} \\ &< \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.\end{aligned} \tag{4.44}$$

Hence, by (2.9) of Lemma 2.4, we have that

$$\|(T_{x_0}+A)^{-1}F'(x_0)\| \le \frac{1}{1-\|T_{x_0}^{-1}A\|} \le \frac{1}{2-\|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2} < 2. \tag{4.45}$$

Noting $\tilde{x}_0 \in \mathbb{B}(x_0, \frac{3-2\sqrt{2}}{\gamma+\epsilon})$ and $\|T_{x_0}^{-1}A\| < \frac{1}{2}$ by (4.43) and (4.44), respectively, one can apply (4.8) to see that

$$(T_{x_0}+A, F_{E_2}) \text{ satisfies the weak } \tilde{\gamma}\text{-condition on } \mathbb{B}\left(\tilde{x}_0, \frac{2-\sqrt{2}}{2\tilde{\gamma}}\right). \tag{4.46}$$

Recalling the definition of $\tilde{\xi}$ given in (4.42), one has from (2.7) and (4.45) that

$$\tilde{\xi} \le \|(T_{x_0}+A)^{-1}F'(x_0)\| \cdot \|T_{x_0}^{-1}(-F_{E_2}(\tilde{x}_0))\| \le 2\|T_{x_0}^{-1}(-F_{E_2}(\tilde{x}_0))\| \le +\infty; \tag{4.47}$$

similarly, (2.7) and (4.45) imply that

$$\|(T_{x_0}+A)^{-1}\|_{\mathrm{d}} \le \frac{\|T_{x_0}^{-1}\|_{\mathrm{d}}}{2-\|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2} \le 2\|T_{x_0}^{-1}\|_{\mathrm{d}} < +\infty, \tag{4.48}$$

where the last inequality holds by the last line of (3.1). Moreover, $\mathrm{R}(T_{x_0}) = \mathrm{R}(T_{x_0}+A)$ (by (2.9) and (4.44)), and so (3.14) implies that

$$\operatorname{srec} K \cap \mathrm{R}(T_{x_0}+A) \ne \emptyset. \tag{4.49}$$

Assuming

$$\tilde{\gamma}\tilde{\xi} < \frac{\tau-1}{\tau(2\tau-1)}, \tag{4.50}$$

with $\tilde{\gamma}$, $\tilde{\xi}$ being defined as in (4.9) and (4.42), and $\tau \in (1, \frac{2+\sqrt{2}}{2}]$, we will verify that

$$\mathrm{d}(\tilde{x}_0, \mathcal{S}(E_2)) \le \frac{\tau\,\|T_{x_0}^{-1}\|_{\mathrm{d}}\big[\mathrm{d}(F_{E_2}(\tilde{x}_0), C\cap(F_{E_2}(\tilde{x}_0)+\mathrm{R}(T_{x_0})))\big]}{2-\|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2} < +\infty. \tag{4.51}$$

Indeed, by (4.50) and (3.4) (applied to $\tilde{\gamma}$, $\tilde{\xi}$ in place of γ, ξ), one has $\tilde{r}^* < \frac{2-\sqrt{2}}{2\tilde{\gamma}}$, where

$$\tilde{r}^* := \frac{2\tilde{\xi}}{1+\tilde{\gamma}\tilde{\xi}+\sqrt{(1+\tilde{\gamma}\tilde{\xi})^2 - 8\tilde{\gamma}\tilde{\xi}}}$$

is the number defined in (3.3) with $\tilde{\gamma}$, $\tilde{\xi}$ in place of γ, ξ. Noting that $T_{x_0}+A$ is just the convex process at $\tilde{x}_0$ as defined in (2.5) associated with the pair (C, F_{E_2}) in place of (C, F), it follows that (3.15) of Theorem 3.3 is applicable to $\tilde{x}_0,\ F_{E_2},\ \tilde{\gamma},\ \frac{2-\sqrt{2}}{2\tilde{\gamma}},\ \tilde{r}^*$, $\mathcal{S}(E)$ in place of $x_0,\ F,\ \gamma,\ \Lambda,\ r^*,\ S$ (thanks to (4.46), (4.48), (4.49), and (4.50)), and so

$$\mathrm{d}(\tilde{x}_0, \mathcal{S}(E_2)) \le \tau \|(T_{x_0}+A)^{-1}\|_{\mathrm{d}} \mathrm{d}(F_{E_2}(\tilde{x}_0), C \cap (F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0}+A))). \tag{4.52}$$

Thus (4.51) holds (assuming (4.50)) by inequality (4.48) and the fact that $\mathrm{R}(T_{x_0}) = \mathrm{R}(T_{x_0}+A)$. The rest of our proof is divided into two parts dealing with (i) and (ii), respectively. For this purpose, we first note from (4.9) that

$$\tilde{\gamma} = \frac{2(\gamma+\epsilon)}{(1-(\gamma+\epsilon)\|\tilde{x}_0 - x_0\|)^3} \le \frac{2(\gamma+\epsilon)}{(1-\frac{2-\sqrt{3}}{2})^3} = \frac{2(\gamma+\epsilon)}{(\sqrt{3}/2)^3} < 2\sqrt{3}(\gamma+\epsilon), \tag{4.53}$$

because $\|\tilde{x}_0 - x_0\| < r \le \frac{2-\sqrt{3}}{2(\gamma+\epsilon)}$ (see (4.43)).

(i) Since $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$, we take δ_0 such that $0 < \delta_0 \le r$ and

$$(2+L)\delta_0 \mathbb{B} \subseteq F(x_0) + \mathrm{R}(T_{x_0}), \tag{4.54}$$

where $L \ge 0$ is a Lipschitz constant of F on $\mathbb{B}(x_0, r_0)$. Let constants $\mu_0,\ \mu,\ \delta$ be defined by

$$\begin{aligned} \mu_0 &:= 1 + 4(1+L) + \frac{4\|F(x_0)\|}{\delta_0}, \quad \mu := \frac{(2+\sqrt{2})\mu_0}{2}, \\ \delta &:= \min\left\{\delta_0,\ \frac{3-2\sqrt{2}}{16\sqrt{3}\mu_0\|T_{x_0}^{-1}\|_{\mathrm{d}}(\gamma+\epsilon)}\right\}. \end{aligned} \tag{4.55}$$

Below we show that the pair (δ, μ) is as desired. To do this, let $|||E_i|||_r < \delta$ satisfy (4.37). To show (4.36), assuming $\tilde{x}_0 \in \mathbb{B}(x_0, \delta) \cap \mathcal{S}(E_1)$, we have to show that

$$\mathrm{d}(\tilde{x}_0, \mathcal{S}(E_2)) \le \frac{(2+\sqrt{2})\mu_0 \|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r}{2(2 - \|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2)}. \tag{4.56}$$

To this end, note first that

$$\|F_{E_2}(\tilde{x}_0) - F(x_0)\| \le L\|\tilde{x}_0 - x_0\| + |||E_2|||_r \le (1+L)\delta, \tag{4.57}$$

and it follows from (4.54) that $\delta_0\mathbb{B} \subseteq F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0})$ (noting that $\delta \le \delta_0$), and this, in particular, implies $-F_{E_2}(\tilde{x}_0) \in \mathrm{R}(T_{x_0})$. Thus by (4.22) of Lemma 4.5 (applied to $F_{E_2}(\tilde{x}_0),\ C,\ F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0})$ in place of $y,\ D,\ Z$), and noting $F_{E_2}(\tilde{x}_0) \in F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0})$, we have that

$$\begin{aligned} \mathrm{d}(F_{E_2}(\tilde{x}_0), C \cap (F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0}))) &\le \left(1 + \frac{4\|F_{E_2}(\tilde{x}_0)\|}{\delta_0}\right) \mathrm{d}(F_{E_2}(\tilde{x}_0), C) \\ &\le \mu_0 |||E_1 - E_2|||_r, \end{aligned} \tag{4.58}$$

where the last inequality holds because, by $F_{E_1}(\tilde{x}_0) \in C$ (as $\tilde{x}_0 \in \mathcal{S}(E_1)$) and $\tilde{x}_0 \in \mathbb{B}(x_0, r)$,

$$\mathrm{d}(F_{E_2}(\tilde{x}_0), C) \le \|F_{E_2}(\tilde{x}_0) - F_{E_1}(\tilde{x}_0)\| \le \|E_1(\tilde{x}_0) - E_2(\tilde{x}_0)\| \le |||E_1 - E_2|||_r,$$

and, by (4.57) and (4.55),

$$1 + \frac{4\|F_{E_2}(\tilde{x}_0)\|}{\delta_0} \le 1 + 4(1+L) + \frac{4\|F(x_0)\|}{\delta_0} = \mu_0.$$

Thus, as noted earlier, $-F_{E_2}(\tilde{x}_0) \in \mathrm{R}(T_{x_0})$, and so we can apply (2.8) (to $F_{E_2}(\tilde{x}_0)$ in place of y), and so

$$\begin{aligned} \|T_{x_0}^{-1}(-F_{E_2}(\tilde{x}_0))\| &\le \|T_{x_0}^{-1}\|_{\mathrm{d}} \mathrm{d}(F_{E_2}(\tilde{x}_0), C \cap (F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0}))) \\ &\le \mu_0 \|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r. \end{aligned} \tag{4.59}$$

Hence, by (4.47), we have that

$$\tilde{\xi} \le 2\mu_0 \|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r \le 4\delta\mu_0 \|T_{x_0}^{-1}\|_{\mathrm{d}}, \tag{4.60}$$

and it follows from (4.53) and (4.55) that

$$\tilde{\gamma}\tilde{\xi} \le 4\mu_0\tilde{\gamma}\|T_{x_0}^{-1}\|_{\mathrm{d}}\delta \le \frac{\tilde{\gamma}(3-2\sqrt{2})}{4\sqrt{3}(\gamma+\epsilon)} < 3 - 2\sqrt{2}.$$

Therefore (4.50) holds (and so does (4.51)) with τ replaced by $\frac{2+\sqrt{2}}{2}$ (so the right-hand side of (4.50) is $3 - 2\sqrt{2}$; see (3.9)). It follows from (4.51) and (4.58) that

$$\mathrm{d}(\tilde{x}_0, \mathcal{S}(E_2)) \le \frac{2+\sqrt{2}}{2} \cdot \|T_{x_0}^{-1}\|_{\mathrm{d}} \cdot \frac{\mu_0 |||E_1 - E_2|||_r}{2 - \|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2},$$

verifying (4.56).

(ii) Assume that (4.37), (4.38) hold and $E_1(x_0) - E_2(x_0) \in \mathrm{R}(T_{x_0})$. Clearly, $F_{E_1}(x_0) - F_{E_2}(x_0) = E_1(x_0) - E_2(x_0) \in \mathrm{R}(T_{x_0})$ and $\mathrm{R}(F_{E_i}'(x_0)) \subseteq \mathrm{R}(T_{x_0})$ (see (4.37) and note that $\mathrm{R}(F'(x_0)) \subseteq \mathrm{R}(T_{x_0})$). Hence, making use of (4.39) and (4.43), we can apply Lemma 4.9 to $F_{E_1},\ F_{E_2},\ \gamma+\epsilon,\ \tilde{x}_0$ in place of $F_1,\ F_2,\ \bar{\gamma},\ x$, and so (4.34) tells us that $F_{E_1}(\tilde{x}_0) - F_{E_2}(\tilde{x}_0) \in \mathrm{R}(T_{x_0})$, that is $F_{E_1}(\tilde{x}_0) \in F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0})$. Also, since $F_{E_1}(\tilde{x}_0) \in K \subseteq C$ as $\tilde{x}_0 \in \mathcal{S}(E_1)$ by (4.41), it follows that

$$\mathrm{d}(F_{E_2}(\tilde{x}_0), C \cap (F_{E_2}(\tilde{x}_0) + \mathrm{R}(T_{x_0})) \le \|F_{E_1}(\tilde{x}_0) - F_{E_2}(\tilde{x}_0))\| \le |||E_1 - E_2|||_r. \tag{4.61}$$

Furthermore, the first inequality of (4.59) in the above proof of (i) is still valid for the present case (since, as noted earlier, $-F_{E_2}(\tilde{x}_0) \in -F_{E_1}(\tilde{x}_0) + \mathrm{R}(T_{x_0}) \subseteq \mathrm{R}(T_{x_0})$, and thus (2.8) is applicable). Therefore, it follows from (4.47), (4.61), and the first inequality of (4.59) that

$$\tilde{\gamma}\tilde{\xi} \le 2\tilde{\gamma}\|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r \le 2\tilde{\gamma}\|T_{x_0}^{-1}\|_{\mathrm{d}}(|||E_1|||_r + |||E_2|||_r) < \frac{\tilde{\gamma}(\tau-1)}{2\sqrt{3}(\gamma+\epsilon)\tau(2\tau-1)}, \tag{4.62}$$

where the last inequality holds by the assumed (4.38), and therefore (4.50) holds by (4.53) for the present case. Consequently, (4.51) is applicable and, together with (4.61), one has that

$$\mathrm{d}(\tilde{x}_0, \mathcal{S}(E_2)) \le \frac{\tau \|T_{x_0}^{-1}\|_{\mathrm{d}} |||E_1 - E_2|||_r}{2 - \|T_{x_0}^{-1}E_2'(x_0)\| - 1/(1-(\gamma+\epsilon)r)^2},$$

and so (4.36) is shown with $\delta = r$ and $\mu = \tau$ as X is reflexive and $\tilde{x}_0 \in \mathbb{B}(x_0, r) \cap \mathcal{S}(E_1)$ is arbitrary. The proof is complete. ∎

COROLLARY 4.11. *Assume that* (3.1), (3.14), $\Lambda = r_0 := \frac{2-\sqrt{2}}{2\gamma}$ *hold and suppose that* $x_0 \in S_{\mathrm{cl}}$ *is such that* (T_{x_0}, F) *satisfies* (3.20). *Let* $\epsilon \in [0, +\infty)$, $r \in (0, \frac{2-\sqrt{3}}{2(\gamma+\epsilon)})$, $\tau \in (1, \frac{2+\sqrt{2}}{2}]$, *and* $E_1, E_2 \in \mathcal{H}_0^2(x_0; r)$ *be such that* (4.37) *and* (4.38) *hold for each* $i = 1, 2$. *Then* (4.36) *holds with* $\delta = r$ *and* $\mu = \tau$, *and*

$$\overline{\mathbb{B}}\left(x_0, \frac{\tau\, \|T_{x_0}^{-1}\|_{\mathrm{d}} \|E_i(x_0)\|}{1 - \|T_{x_0}^{-1}E_i'(x_0)\|}\right) \cap \mathcal{S}(E_i) \ne \emptyset \quad \textit{for each } i = 1, 2. \tag{4.63}$$

Proof. To verify (4.63), it suffices to show that Corollary 4.4 is applicable to each E_i $(i = 1, 2)$ in place of E. Indeed, thanks to the assumptions $E_i \in \mathcal{H}_0^2(x_0; r)$ and $x_0 \in S_{\rm cl}$, we know that $-E_i(x_0) \in \mathrm{R}(T_{x_0})$ and $\|T_{x_0}^{-1}(-F(x_0))\| = 0$. This and the triangle inequality, together with (4.38), imply that

$$\|T_{x_0}^{-1}(-(F + E_i)(x_0))\| \le \|T_{x_0}^{-1}(E_i(x_0))\| < \frac{\tau - 1}{8\sqrt{3}\tau(\gamma + \epsilon)(2\tau - 1)}$$

and so E_i satisfies (4.13) stated for E. It also satisfies (4.6) by (4.37). Therefore (4.63) holds by Corollary 4.4.

To show (4.36) (with $\mu = \tau$), we only need to justify applying Theorem 4.10(ii) to the present pair E_1, E_2, namely to show $E_1(x_0) - E_2(x_0) \in \mathrm{R}(T_{x_0})$. To do this, take $\tilde{x}_0 \in \mathbb{B}(x_0, r) \cap \mathcal{S}(E_1)$. Note that $F_{E_1}(\tilde{x}_0) \in K \subseteq \mathrm{R}(T_{x_0})$ by (3.22) (thanks to (3.20)), and $\tilde{x}_0 \in \mathbb{B}(x_0, \frac{2-\sqrt{3}}{2(\gamma+\epsilon)})$ since $r < \frac{2-\sqrt{3}}{2(\gamma+\epsilon)}$ by assumption. Thus, one applies (4.34) of Lemma 4.9 (to F_{E_1}, 0 in place of F_1, F_2) to check that $F_{E_1}(x_0) \in \mathrm{R}(T_{x_0})$. Since $-F(x_0) \in -C \subseteq \mathrm{R}(T_{x_0})$ (as $x_0 \in S_{\rm cl}$) and $-E_2(x_0) \in \mathrm{R}(T_{x_0})$, it follows that $-F_{E_2}(x_0) = -F(x_0) - E_2(x_0) \in \mathrm{R}(T_{x_0})$. Hence $E_1(x_0) - E_2(x_0) = F_{E_1}(x_0) - F_{E_2}(x_0) \in \mathrm{R}(T_{x_0})$ as desired. The proof is complete. ∎

Recall that we introduced in Definition 4.1 the exact Lipschitzian bounds for set-valued mappings.

THEOREM 4.12. *Assume that* (3.1), (3.14) *hold and* $\Lambda = r_0$. *Let* $x_0 \in S$ *and* $R \in (0, r_0]$.

(i) *Suppose* $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$. *Then the solution map* $\mathcal{S} : \mathcal{H}^2(x_0; R) \rightrightarrows X$ *is Lipschitz-like around* $(0, x_0)$.

(ii) *Suppose that* (3.20) *holds. Then the solution map* $\mathcal{S} : \mathcal{H}_0^2(x_0; R) \rightrightarrows X$ *is Lipschitz-like around* $(0, x_0)$ *and the exact Lipschitzian bound satisfies* $\mathrm{lip}_{\mathcal{H}_0^2}\mathcal{S}(0, x_0) \le \|T_{x_0}^{-1}\|_{\rm d}$

Proof. Similar to the discussion in the proof for Theorem 4.7, we assume, without loss of generality, that $R = r_0 = \frac{2-\sqrt{2}}{2\gamma}$. Let $0 < \epsilon \le \frac{1}{6}$, $1 < \tau \le \frac{2+\sqrt{2}}{2}$, and $0 < r \le \min\{\frac{1}{2}, \frac{2-\sqrt{3}}{2(\gamma+\epsilon)}\}$. We select a pair (δ_τ, μ_τ) of two positive numbers according to the following rules:

(a) If $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$ but $F(x_0) \notin \mathrm{R}(T_{x_0})$, then we take our (δ_τ, μ_τ) to be the pair (δ, μ) as in (i) of Theorem 4.10.

(b) If $F(x_0) \in \mathrm{R}(T_{x_0})$, then we take $(\delta_\tau, \mu_\tau) := (\delta, \tau)$ with $\delta \in (0, r)$.

Since $r \le \frac{1}{2}$ and $r \le \frac{2-\sqrt{3}}{2(\gamma+\epsilon)} < R$, it follows from (4.4) that

$$|||E|||_r \le (1 + r + r^2)\|E\|_r \le (1 + 2r)\|E\|_R \quad \text{for each } E \in \mathcal{C}^2(\mathbb{B}(x_0, R), Y). \tag{4.64}$$

Now set

$$\theta := \min\left\{\frac{\delta_\tau}{2}, \frac{\epsilon}{\|T_{x_0}^{-1}\|_{\rm d}}, \frac{\tau - 1}{8\sqrt{3}(1 + 2r)\|T_{x_0}^{-1}\|_{\rm d}(\gamma + \epsilon)\tau(2\tau - 1)}\right\} \tag{4.65}$$

and let $E_1, E_2 \in V := \{E \in \mathcal{H}^2(x_0; R) : \|E\|_R < \theta\}$. Fix $i = 1, 2$, and note that

$$\|T_{x_0}^{-1}E_i'(x_0)\| \le \|T_{x_0}^{-1}\|_{\rm d}\|E_i'(x_0)\| \le \|T_{x_0}^{-1}\|_{\rm d}\|E_i\|_R < \epsilon \le \frac{1}{6}, \tag{4.66}$$

and that, similarly,

$$\|T_{x_0}^{-1}E_i''(x)\| \le \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E_i\|_R < \epsilon < \frac{2\epsilon}{(1-\epsilon\|x-x_0\|)^3} \quad \text{for each } x \in \mathbb{B}\left(x_0, \frac{2-\sqrt{2}}{2(\gamma+\epsilon)}\right),$$

and

$$\|T_{x_0}^{-1}\|_{\mathrm{d}}|||E_i|||_r \le (1+2r)\|T_{x_0}^{-1}\|_{\mathrm{d}}\|E_i\|_R < \frac{\tau-1}{8\sqrt{3}(\gamma+\epsilon)\tau(2\tau-1)}.$$

Thus, (4.37) and (4.38) of Theorem 4.10 are checked. Write $r_\tau := \min\{\delta_\tau, r\}$. Below we will show that

$$\mathbb{B}(x_0, r_\tau) \cap \mathcal{S}(E_1) \subseteq \mathcal{S}(E_2) + \frac{\mu_\tau \|T_{x_0}^{-1}\|_{\mathrm{d}}|||E_1-E_2|||_r}{2-\epsilon-1/(1-(\gamma+\epsilon)r)^2}\overline{\mathbb{B}} \tag{4.67}$$

for each of the cases in (i) and (ii). Granting this and making use of the fact from (4.64) that $|||(E_1-E_2)|||_r \le (1+2r)\|(E_1-E_2)\|_R$, we have that

$$\mathbb{B}(x_0, r_\tau) \cap \mathcal{S}(E_1) \subseteq \mathcal{S}(E_2) + \frac{(1+2r)\mu_\tau \|T_{x_0}^{-1}\|_{\mathrm{d}}\|E_1-E_2\|_R}{2-\epsilon-1/(1-(\gamma+\epsilon)r)^2}\overline{\mathbb{B}}. \tag{4.68}$$

(i) $-F(x_0) \in \mathrm{int}(\mathrm{R}(T_{x_0}))$. Without loss of generality, we may assume that $F(x_0) \notin \mathrm{R}(T_{x_0})$ (otherwise, $0 \in \mathrm{int}(\mathrm{R}(T_{x_0}))$; hence $\mathrm{R}(T_{x_0}) = Y$ and (3.20) holds). Then, noting $r_\tau \le \delta_\tau = \delta$ and $\|T_{x_0}^{-1}E_i'(x_0)\| < \epsilon$ (by (4.66)), we have by Theorem 4.10(i) that (4.67) and (4.68) hold for all $E_1, E_2 \in V$; consequently, $\mathcal{S} : \mathcal{H}^2(x_0; R) \rightrightarrows X$ is Lipschitz-like around $(0, x_0)$.

(ii) $E_1, E_2 \in \mathcal{H}_0^2(x_0; R)$ and (3.20) holds. Then, thanks to the assumption $x_0 \in S$, $F(x_0) \in C \subseteq \mathrm{R}(T_{x_0})$ (noting (3.22)), and so $\mu_\tau = \tau$ by our selection in (b). Therefore, as noted earlier, we can apply Corollary 4.11 to see that (4.67) and (4.68) hold with $\mu_\tau = \tau$ because $r_\tau \le \delta_\tau = \delta$ and $\|T_{x_0}^{-1}E_i'(x_0)\| < \epsilon$ (see (4.66)). Thus the proof can be completed as in (i). ∎

As noted earlier, (3.14) holds if K is closed. Thus we have the following corollary.

Corollary 4.13. *Assume that* (3.1) *holds and* $\Lambda = r_0$. *Suppose that* $x_0 \in S$ *and let* $R \in (0, r_0]$. *If* K *is closed, then conclusions* (i) *and* (ii) *in Theorem* 4.12 *hold.*

Under the Robinson condition made in [34], T_{x_0} is surjective and we have the following Lipschitz-like property for the solution map, which is a direct consequence of Theorem 4.12.

Corollary 4.14. *Assume that* (3.1) *holds and* $\Lambda = r_0$. *Let* $\gamma \in (0, +\infty)$ *and* $x_0 \in S$ *be such that* T_{x_0} *is surjective. Suppose further that* $\mathrm{srec}\, K \ne \emptyset$. *Then, for any* $R \in (0, r_0]$, *the solution map* $\mathcal{S} : \mathcal{C}_R^2(\mathbb{B}(x_0, r_0), Y) \rightrightarrows X$ *is Lipschitz-like around* $(0, x_0)$ *and the exact Lipschitzian bound satisfies* $\mathrm{lip}_{\mathcal{C}_R^2}\mathcal{S}(0, x_0) \le \|T_{x_0}^{-1}\|_{\mathrm{d}} < +\infty$.

Remark 4.1. Considering the special case in which F and the perturbation are affine, the results of Theorems 4.7 and 4.12 are new for the abstract inequality system (1.1) in the case in which either K is not closed (such as for the inequality/strict-inequality system (1.4) with nonempty I_P) or X is of infinite dimension. Even when K is closed (such as for the inequality/strict-inequality system (1.4) with empty I_P) and $X = \mathbb{R}^n$, part (ii) of Theorem 4.12 is new, while part (i) is essentially known as it can be deduced from the corresponding results in [26]. Moreover, for the finite/infinite inequality system (1.3), part (ii) of Theorems 4.7 and 4.12 is new while part (i) is essentially known as it can be deduced from the corresponding results in [5, 11, 18].

We end our paper with two examples illustrating the use of Theorem 4.12 and Corollary 4.14 for showing the lower semicontinuity and/or the Lipschitz-like property of the concerned solution map.

Example 4.1. Let $X=\mathbb{R}^2$, $Y=\mathbb{R}^3$, and let the "cone" $K\subseteq\mathbb{R}^3$ be given by

$$K:=\{(t_1,t_2,t_3):\ t_1^2+(t_3-t_2)^2\le t_2^2 \text{ and } t_2<0\},$$

that is, $C=\overline{K}$ is the closed cone generated by the origin and the plane disk $\{(t_1,-1,t_3):\ t_1^2+(t_3+1)^2\le 1\}$ while $K=C\setminus\{0\}$. Let $(\lambda_1,\lambda_2,\lambda_3)\in K$, and define F by

$$F(x):=\begin{pmatrix} t_1^2+t_2+\lambda_1\\ t_1+\frac{t_2^2}{1-t_2}+\lambda_2\\ +\lambda_3\end{pmatrix}\quad\text{for each } x=(t_1,t_2)\in\mathbb{R}\times(-\infty,1).$$

Then

$$F'(x)=\begin{pmatrix} 2t_1 & 1\\ 1 & -1+\frac{1}{(1-t_2)^2}\\ 0 & 0\end{pmatrix}\quad\text{for each } x=(t_1,t_2)\in\mathbb{R}\times(-\infty,1).$$

Let $x_0:=0$. Then $F(x_0)\in K\subseteq C$ (so $x_0\in S$ and $\xi=0$), and also

$$F'(x_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\\ 0 & 0\end{pmatrix}.$$

Hence

$$T_{x_0}u=(u_2,u_1,0)^T-C\quad\text{for each } u=(u_1,u_2)\in\mathbb{R}^2.$$

This implies that $\mathrm{R}(T_{x_0})=\mathbb{R}^2\times\mathbb{R}_+$. Hence, (3.14) holds (as $(0,-1,0)\in \mathrm{srec}K\cap \mathrm{R}(T_{x_0})$). Moreover, for any $z=(z_1,z_2,z_3)\in\mathbb{R}^2\times\mathbb{R}_+$, one sees that

$$T_{x_0}^{-1}z=\{(u_1,u_2):(u_2-z_1)^2+(z_3+(u_1-z_2))^2\le(u_1-z_2)^2 \text{ and } u_1\le z_2\};$$

hence $(z_2-z_3,z_1)\in T_{x_0}^{-1}(z_1,z_2,z_3)$. Thus we have that

$$\|T_{x_0}^{-1}z\|\le\sqrt{z_1^2+(z_2-z_3)^2}\le\sqrt{2}\|z\|\quad\text{for each } z=(z_1,z_2,z_3)\in\mathbb{R}^2\times\mathbb{R}_+. \tag{4.69}$$

This immediately yields that $\|T_{x_0}^{-1}\|\le\sqrt{2}$. Now let $x=(t_1,t_2)\in\mathbb{R}\times(-\infty,1)$, and $u=(u_1,u_2)$, $v=(v_1,v_2)\in\mathbb{B}$. One checks by definition that

$$F''(x)(u,v)=\begin{pmatrix} u\begin{pmatrix} 2 & 0\\ 0 & 0\end{pmatrix}v^T\\ u\begin{pmatrix} 0 & 0\\ 0 & \frac{2}{(1-t_2)^3}\end{pmatrix}v^T\\ 0\end{pmatrix}=\begin{pmatrix} 2u_1v_1\\ \frac{2u_2v_2}{(1-t_2)^3}\\ 0\end{pmatrix}.$$

Therefore, $\mathrm{R}(F''(x)) \subseteq \mathrm{R}(T_{x_0})$, and, thanks to (4.69),

$$\|T_{x_0}^{-1}F''(x)(u,v)\| \leq \sqrt{2}\sqrt{(2u_1v_1)^2 + \frac{(2u_2v_2)^2}{(1-t_2)^6}} \leq \frac{2\sqrt{2}(|u_1v_1| + |u_2v_2|)}{(1-t_2)^3} \leq \frac{2\sqrt{2}}{(1-t_2)^3}$$

(noting that $|u_1v_1| + |u_2v_2| \leq \frac{1}{2}(u_1^2 + v_1^2 + u_2^2 + v_2^2) \leq 1$); hence we have in particular that

$$\|T_{x_0}^{-1}F''(x)\| \leq \frac{2\sqrt{2}}{(1-t_2)^3} \leq \frac{2\sqrt{2}}{(1-\sqrt{2}\|x\|)^3} \quad \text{for each } x \in \mathbb{B}\left(0, \tfrac{1}{\sqrt{2}}\right).$$

Therefore, assumption (3.1) holds with $\gamma = \sqrt{2}$ and $\Lambda = r_0 = (2-\sqrt{2})/2\sqrt{2}$ (see (3.3)). Let $R \in (0, (2-\sqrt{2})/2\sqrt{2}]$. Thus Theorem 4.7 is applicable and the solution map $\mathcal{S}(\cdot) : \mathcal{H}_0^2(x_0; R) \rightrightarrows X$ is lower semicontinuous at $(0,0)$. Furthermore, if it is assumed that $\lambda_3 < 0$ then $-F(x_0) \in \mathrm{int}\mathrm{R}(T_{x_0})$, Theorem 4.12 is applicable to getting that the solution map $\mathcal{S} : \mathcal{H}^2(x_0; R) \rightrightarrows X$ is Lipschitz-like around $(0,0)$.

For the following example, let $\{f_i : i \in \mathbb{N}\}$ be a pointwise bounded family $\{f_i : i \in \mathbb{N}\}$ in $\mathcal{C}^2(\mathbb{B}, \mathbb{R})$, and we need some general observations on the family $\{f_i : i \in \mathbb{N}\}$. Let l^∞ denote the classical Banach space consisting of all bounded sequences of real numbers endowed with the supremum norm given by

$$\|y\| := \sup_{i\in\mathbb{N}} |t_i| \quad \text{for each } y := (t_i) \in l^\infty.$$

Define $F : \mathbb{B} \to l^\infty$ by

$$F(x) := (f_i(x)) \quad \text{for each } x \in \mathbb{B}. \tag{4.70}$$

Fixing $\bar{x} \in \mathbb{B}$ and assuming that $\{f_i'' : i \in \mathbb{N}\}$ is equicontinuous around $\bar{x}$, one can verify by definition that F is twice differentiable at $\bar{x}$, and the first and the second derivatives are given by

$$F'(\bar{x})u = (f_i'(\bar{x})u) \quad \text{for each } u \in X \tag{4.71}$$

and

$$F''(\bar{x})(u,v) = (f_i''(\bar{x})(u,v)) \quad \text{for each } (u,v) \in X \times X,$$

respectively. In particular, if $\{f_i'' : i \in \mathbb{N}\}$ is equicontinuous on any closed subset of $\mathbb{B}$, then $F \in \mathcal{C}^2(\mathbb{B}, l^\infty)$. Furthermore, we have that

$$\|F''(\bar{x})\| \leq \sup_{i\in\mathbb{N}} \|f_i''(\bar{x})\|.$$

In Example 4.2 below, we tailor (1.4) to the case in which $I := \mathbb{N}$, $I_P := \mathbb{N}_1$ consists of all odd natural numbers, $I_N := \mathbb{N}_2$ consists of all even natural numbers, and I_E is the empty set, namely the following infinite inequality/strict-inequality system:

$$\begin{aligned} f_n(x) &> 0, \qquad n \in \mathbb{N}_1, \\ f_n(x) &\geq 0, \qquad n \in \mathbb{N}_2. \end{aligned} \tag{4.72}$$

We pick and fix a sequence $\{A_n : n \in \mathbb{N}\}$ of symmetric matrices and $(\lambda_n) \in l^\infty$ such that

$$A_n \in \mathbb{R}^{n\times n}, \;\; \|A_n\| \leq 1 \quad \text{for each } n \in \mathbb{N}, \tag{4.73}$$

and

$$\lambda_n \geq 0 \text{ for each } n \in \mathbb{N}_1 \quad \text{and} \quad \lambda_n > 0 \text{ for each } n \in \mathbb{N}_2. \tag{4.74}$$

Let X denote the Hilbert space l^2 consisting of all square-summable sequences of real numbers. For each $x := (t_n) \in X$ and $k \in \mathbb{N}$, we write $x|^k$ for $(t_1, \dots, t_k) \in \mathbb{R}^k$.

Example 4.2. For each $n \in \mathbb{N}$, let $f_n : X \to R$ be defined by

$$f_n(x) := \lambda_n + t_1 + t_n + \frac{1}{2}\langle A_n x|^n, x|^n\rangle + \sum_{k=1}^{n+1} \frac{t_{n+1}^{k+1}}{2(k+1)} \quad \text{for each } x = (t_n) \in X.$$

Then, fixing $x = (t_n) \in X$, one checks by definition that

$$f_n'(x)u = u_1 + u_n + \langle A_n x|^n, u|^n\rangle + \sum_{k=1}^{n+1} \frac{t_{n+1}^{k}}{2} u_{n+1} \quad \text{for each } u = (u_n) \in X, \tag{4.75}$$

and

$$f_n''(x)(u,v) = \langle A_n u|^n, v|^n\rangle + \sum_{k=1}^{n+1} \frac{k}{2} t_{n+1}^{k-1} u_{n+1} v_{n+1} \quad \text{for each } (u,v) = ((u_n),(v_n)) \in X^2.$$

Then it is trivial to check by elementary calculus that $\{f_i'' : i \in \mathbb{N}\}$ is equicontinuous on any closed subset of $\mathbb{B}$ and, for each $x \in \mathbb{B}$,

$$\|f_n''(x)\| \leq \|A_n\| + \sum_{k=1}^{\infty} \frac{k}{2} |t_{n+1}|^{k-1} \leq 1 + \frac{1}{(1-|t_{n+1}|)^3} \leq \frac{2}{(1-\|x\|)^3}.$$

Thus the function F defined by (4.70) is in $\mathcal{C}^2(\mathbb{B}, l^\infty)$, and

$$\|F''(x)\| \leq \sup_{i \in I} \|f_i''(x)\| \leq \frac{2}{(1-\|x\|)^3} \quad \text{for each } x \in \mathbb{B}.$$

Let $K \subseteq l^\infty$ be the "cone" consisting of all sequences $(s_n) \in l^\infty$ such that $s_n \geq 0$ if $n \in \mathbb{N}_1$ and $s_n > 0$ otherwise. Then $\mathrm{srec} K \neq \emptyset$, and the infinite inequality/strict-inequality (4.72) coincides with (1.1). Let $x_0 := 0$. Then $x_0 \in S$, the solution set of (4.72) (noting that $F(x_0) = (\lambda_n) \in K$), and $\xi = 0$. Since by (4.71) and (4.75)

$$F'(x_0)u := (f_n'(x_0)u) = (u_1 + u_n) \quad \text{for each } u = (u_n) \in X,$$

it follows that T_{x_0} is surjective and $\|T_{x_0}^{-1}\| = 1$. Indeed, for any $z := (z_n) \in l^\infty$ with $\|z\| = 1$, one has that

$$T_{x_0}^{-1} z = \left\{(u_n) \in X : u_1 \geq \frac{z_1}{2},\ u_n \geq z_n - u_1 \ \forall n \geq 2\right\};$$

hence $(1, 0, \dots) \in T_{x_0}^{-1} z$ and $\|T_{x_0}^{-1}\| \leq 1$. On the other hand, for $\bar{z} := (1)$, we have that

$$T_{x_0}^{-1} \bar{z} = \{(u_1, 0, \dots) : u_1 \geq 1\}$$

and so $\|T_{x_0}^{-1}\| \geq \|T_{x_0}^{-1}\bar{z}\| = 1$. Therefore, assumption (3.1) holds with $\gamma = 1$ and $\Lambda = r_0 = \frac{2-\sqrt{2}}{2}$. Thus, Corollary 4.14 is applicable, and, for any $R \in (0, \frac{2-\sqrt{2}}{2})$, the solution map $\mathcal{S} : \mathcal{C}_R^2(\mathbb{B}(x_0, \frac{2-\sqrt{2}}{2}), l^\infty) \rightrightarrows X$ is Lipschitz-like around $(0,0)$ and the exact Lipschitzian bound satisfies $\mathrm{lip}_{\mathcal{C}_R^2} \mathcal{S}(0, x_0) \leq 1$.

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