SIAM J. Optim.
Vol. 24, No. 1, pp. 154–174

METRIC SUBREGULARITY OF PIECEWISE LINEAR MULTIFUNCTIONS AND APPLICATIONS TO PIECEWISE LINEAR MULTIOBJECTIVE OPTIMIZATION*

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Abstract. In this paper, we consider the metric subregularity property for a piecewise linear multifunction (with respect to a piecewise linear constraint) as well as the weak sharp minimum property for a piecewise linear constrained multiobjective optimization problem. Of these properties we pay special attention to the global ones. We first provide a result on a certain relationship between two nonnegative piecewise linear numerical functions for which the kernel of one of them is contained in the kernel of the other. Using this result, we establish the bounded/global metric subregularity results for a piecewise linear multifunction with respect to a piecewise linear set. As applications, we study the weak sharp minimum property for a piecewise linear constrained multiobjective optimization problem.

Key words. piecewise linearity, metric subregularity, weak sharp minima, multiobjective optimization

AMS subject classifications. 90C26, 90C29, 90C31

DOI. 10.1137/120889502

1. Introduction. In many fields, linearity plays an important role and is well studied even though, in some aspects, linearity is often too restrictive. This paper concerns multifunctions F between Banach spaces X and Y. We say that F is a linear multifunction if its graph $\mathrm{Gr}(F)$ is a (convex) polyhedron (see section 2 for its definition), and that F is a piecewise linear multifunction if $\mathrm{Gr}(F)$ is the union of finitely many polyhedra. In the special case when $\mathrm{Dom}(F) = X$ and F is single valued, the piecewise linearity has been studied by some authors (cf. [6, 7, 9, 12, 14, 15, 20, 26, 28, 33, 36, 40]). In this paper, one of our aims is to study metric subregularity for piecewise linear multifunctions. Given $\bar{y} \in Y$ in the range $\mathrm{Ran}(F)$ of F, recall that (a) F is globally metrically subregular for $\bar{y}$, and that (b) F is locally metrically subregular for $\bar{y}$ if there exist positive constants τ and η such that the inequality

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \tag{1.1}$$

holds whenever, respectively, (a) $x \in X$, and (b) $x \in X$ with $d(\bar{y}, F(x)) < \eta$, where $d(x, A) := \inf\{\|x - a\| : a \in A\}$ and, more generally,

$$d(A, A') := \inf\{\|a - a'\| : a \in A, a' \in A'\}$$

(and so $d(A, A') = \infty$ if A or A' is empty). In connection with some optimization problems (e.g., weak sharp minima for constraint optimization problems), we often need to

*Received by the editors August 28, 2012; accepted for publication (in revised form) November 7, 2013; published electronically January 23, 2014.
http://www.siam.org/journals/siopt/24-1/88950.html

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replace $\bar{y}$ by a set $E \subset Y$ and to take it into consideration of a constraint set $\Gamma \subset X$; in this case one would consider the following corresponding notions for the pair(Γ, E).

DEFINITION 1.1. *F is said to be globally (resp., locally) metrically subregular for (Γ, E) if the inequality*

$$d(x, F^{-1}(E) \cap \Gamma) \leq \tau(d(E, F(x)) + d(x, \Gamma)) \tag{1.2}$$

holds for all $x \in X$ (resp., for all $x \in X$ with $d(E, F(x)) + d(x, \Gamma) < \eta$).

DEFINITION 1.2. *F is said to be boundedly metrically subregular for (Γ, E) if for any $r \in (0, +\infty)$ there exists $\tau \in (0, +\infty)$ such that (1.2) holds for all $x \in X$ with $\|x\| \leq r$.*

The classical error bound theorem of Hoffman [16] (who considered the case when X is finite dimensional) says essentially that if F is a linear multifunction then F is globally metrically subregular (or is of a global error bound) for each $\bar{y} \in \mathrm{Ran}(F)$. Robinson showed in his seminal paper [29] (also assuming that X is finite dimensional) that if F is piecewise linear then F is locally metrically subregular for each $\bar{y} \in \mathrm{Ran}(F)$. Since the global metric subregularity (or error bound property) is clearly more useful than the local one in convergence analysis of optimization, we are naturally led to consider global metric subregularity in the piecewise linear case. One of our main aims is to study global metric subregularity (in the sense that (1.2) holds for some positive constant τ and all x in X) for the case when F is a piecewise linear multifunction and E, Γ are unions of finitely many polyhedra. The metric subregularity and its closely related notions including the metric regularity, linear regularity, error bound, etc., have played important roles with many applications in the development of mathematical programming especially in sensitivity analysis and convergence analysis (see [4, 11, 17, 21, 23, 25, 27, 32, 38, 41]) as well as in multiobjective optimization: Here one assumes that F is a multifunction between two Banach spaces X, Y and $\Gamma \subset X$ with Y being endowed with a partial order (or preorder) defined by a closed convex cone C in Y; the problem is

$$C - \min F(x) \quad \text{subject to} \ \ x \in \Gamma. \tag{1.3}$$

The notion of weak sharp minima is well known in mathematical programming literature (cf. [8, 13, 19, 24, 34]) and has many far-reaching consequences especially in connection with minimizing a proper lower semicontinuous numerical function ϕ: We say that ϕ has the weak sharp minimum property if there exists a positive constant τ such that

$$d(x, S_\phi) \leq \tau(\phi(x) - \lambda) \tag{1.4}$$

for all $x \in X$, where $\lambda = \inf\{\phi(x) : \ x \in X\}$ and $S_\phi = \{x \in X : \ \phi(x) = \lambda\}$. When $\Gamma = X$, $E = \{\lambda\}$, and $F(x) = [\phi(x), +\infty)$, (1.2) reduces to (1.4). If ψ is defined by $\psi(\cdot) = \phi(\cdot) - \lambda$ then that (1.4) holds for all $x \in X$ also means that the inequality $\psi(x) \leq 0$ has a global error bound. For the vector valued situation (F is not necessarily induced by a single-valued function ϕ), the corresponding multiobjective optimization problem has generally many optimal values which may be of different types, e.g., one may take $E = V$ or V_w, where V denotes the set of Pareto optimal values for (1.3), and V_w denotes that of weak-Pareto ones (see section 2 for their definitions). We say that (1.3) has the global/bounded weak sharp minimum property with respect to Pareto optimal values for (1.3) if F is globally/boundedly metrically subregular for (Γ, V). Similarly, we have the corresponding notion for weak-Pareto optimal values. In the case when F is single valued and piecewise linear and under some restrictive assumptions, the weak sharp minimum property of (1.3) has been studied. In particular, Yang and Yen [33] gave a sufficient condition for a single-valued piecewise linear, C-convex function F between two finite dimensional spaces to have a global weak sharp minimum property with respect to Pareto optimal values for (1.3). Some other related results along this line have also been reported in Deng and Yang [10] and Zheng and Yang [39].

The rest of this paper is organized as follows. In section 2, we recall some basic notions and results that are needed later. In section 3, similar to a classical fact on two linear functionals, we prove that if ϕ, ψ are piecewise linear nonnegative real-valued functions on a polyhedron P with $\ker(\phi) := \{x \in P : \ \phi(x) = 0\} \subset \ker(\psi)$ then for any $r > 0$ there exists $\eta > 0$ such that

$$\eta\psi(x) \leq \phi(x) \quad \forall x \in P \cap rB_X$$

and that the above inequality holds for some positive constant η on the entire P if and only if $\lim_{\psi(x)\to\infty} \phi(x) = \infty$. Here and throughout, we adopt the convention that

$$\lim_{\psi(x)\to\infty} \phi(x) = \infty \ \text{ if } \psi \text{ is bounded above.} \tag{1.5}$$

With the help of this result, we prove, under the assumption that E and Γ are unions of finitely many polyhedra, that a piecewise linear multifunction F with $F^{-1}(E) \cap \Gamma \neq \emptyset$ is always locally metrically subregular and boundedly metrically subregular for (Γ, E), and that it is globally metrically subregular for (Γ, E) if and only if $\lim_{d(x, F^{-1}(E)\cap\Gamma)\to\infty} d(F(x), E) + d(x, \Gamma) = \infty$. As applications of these results, we extend some known results on linear regularity from the linear case to the piecewise linear case for a finite collection of sets. Further applications are in section 4 where we establish several sufficient conditions for a multivalued piecewise linear function F (between Banach spaces X and Y) to have global/bounded weak sharp minimum property with respect to Pareto or weak-Pareto optimal values for the corresponding constraint multiobjective problem (1.3).

2. Preliminaries. As two key notions, we first provide the definitions of a polyhedron and a piecewise linear multifunction in a general Banach space case.

Definition 2.1. *A subset P of a Banach space Z is said to be a (convex) polyhedron if there exist $z_1^*, \dots, z_k^* \in Z^*$ and $c_1, \dots, c_k \in \mathbb{R}$ such that*

$$P = \{z \in Z : \ \langle z_i^*, z\rangle \leq c_i, \ i = 1, \dots, k\},$$

where Z^* denotes the dual space of Z.

Definition 2.2. *Let X, Y be Banach spaces and $G : X \rightrightarrows Y$ be a multifunction. We say that G is piecewise linear if there exist finitely many polyhedra $P_1, \dots, P_k$ in the product $X \times Y$ such that $\mathrm{Gr}(G) = \bigcup_{i=1}^k P_i$, where*

$$\mathrm{Gr}(G) := \{(x, y) : \ x \in X \text{ and } y \in G(x)\}$$

is the graph of G. We say that G is linear if $\mathrm{Gr}(G)$ is a polyhedron.

For subspaces Z_1, Z_2 of a linear space Z, we use the notation $Z = Z_1 \oplus Z_2$ to denote that $Z = Z_1 + Z_2$ and $Z_1 \cap Z_2 = \{0\}$. For any subset W of $X \times Y$, let W_X denote its projection on X, namely,

$$W_X := \{x \in X : \ (x, y) \in W \text{ for some } y \in Y\}.$$

We will need the following results on polyhedra, which are known except Lemma 2.2(iii) (see [36, Lemmas 2.1, 2.2, and 2.3]).

LEMMA 2.1. *A subset P of $X \times Y$ is a polyhedron if and only if there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_1 and Y_2 of Y, and a polyhedron P_2 in $X_2 \times Y_2$ such that X_2 and Y_2 are finite dimensional,*

$$X \times Y = (X_1 \times Y_1) \oplus (X_2 \times Y_2) \quad \text{and} \quad P = X_1 \times Y_1 + P_2.$$

LEMMA 2.2. *Let $P \subset X \times Y$, $P_i \subset X$ $(i = 1, 2)$, and $E \subset Y$ be polyhedra. Then the following statements hold:*

(i) *The projection P_X of P on X is a polyhedron;*

(ii) *$P_1 \cap P_2$ and $P_1 + P_2$ are polyhedra;*

(iii) *$P + \{0\} \times E$ is a polyhedron.*

Note that (iii) of Lemma 2.2 follows from (ii) since $P + \{0\} \times E = P + X_1 \times E$, where X_1 is as in Lemma 2.1 (it is easy from Lemma 2.1 to verify that $X_1 \times E$ is a polyhedron in $X \times Y$).

We also need the following lemma, which means that a linear multifunction is globally metrically regular.

LEMMA 2.3. *Let $G : X \rightrightarrows Y$ be a linear multifunction. Then, there exists $\eta \in (0, +\infty)$ such that*

$$\eta d(x, G^{-1}(b)) \leq d(b, G(x)) \ \ \forall x \in X \text{ and } \forall b \in G(X). \tag{2.1}$$

Proof. Define $F : Y \rightrightarrows X$ as follows:

$$F(y) := G^{-1}(y) \quad \forall y \in Y. \tag{2.2}$$

Then $\mathrm{Gr}(F)$ is a polyhedron in $Y \times X$. This and [3, Theorem 2.207] imply that there exists $\eta \in (0, +\infty)$ such that

$$\eta \mathrm{Haus}(F(y), F(b)) \leq \|y - b\| \quad \forall y, b \in \mathrm{dom}(F),$$

where

$$\mathrm{Haus}(F(y), F(b)) := \max\Big\{ \sup_{x \in F(b)} d(x, F(y)), \ \sup_{x \in F(y)} d(x, F(b))\Big\}.$$

Noting that $d(x, F(b)) \leq \mathrm{Haus}(F(y), F(b))$ for all $x \in X$ and $y \in F^{-1}(x)$ and that $d(b, F^{-1}(x)) = +\infty$ if $F^{-1}(x) = \emptyset$, it follows that

$$\eta d(x, F(b)) \leq d(b, F^{-1}(x)) \quad \forall x \in X \text{ and } \ b \in \mathrm{dom}(F).$$

This together with (2.2) implies that (2.1) holds. The proof is completed. ◻

The following lemma is known ([39, Lemma 3.2]) and useful for us.

LEMMA 2.4. *Let P_1 and P_2 be polyhedra in X. Then the following assertions hold:*

(i) *If $P_1 \cap P_2 \neq \emptyset$ then there exists $\eta \in (0, +\infty)$ such that*

$$\eta d(x, P_1 \cap P_2) \leq d(x, P_1) + d(x, P_2) \ \ \forall x \in X;$$

(ii) *if $P_1 \cap P_2 = \emptyset$ then $d(P_1, P_2) > 0$.*

The linearity assumption in Lemma 2.3 and the polyhedron assumption in Lemma 2.4 cannot be dropped even in the finite dimensional case.

The following lemma is an infinite dimensional version of the well-known Hoffman error bound theorem and is essentially known. Using the Hoffman error bound theorem and Lemma 2.1, its proof can be obtained as an easy exercise.

LEMMA 2.5. *Let Y be a normed space, $a_i^* \in Y^*$, and $r_i \in \mathbb{R}$ $(1 \le i \le m)$ and suppose that $Q := \{y \in Y : \langle a_i^*, y\rangle \le r_i,\ 1 \le i \le m\}$ is nonempty. Then there exists $\eta \in (0, +\infty)$ independent of r_i such that*

$$d(b, Q) \le \eta \max_{1\le i\le m} [\langle a_i^*, b\rangle - r_i]_+ \quad \forall b \in Y,$$

where $[\langle a_i^*, b\rangle - r_i]_+ := \max\{0, \langle a_i^*, b\rangle - r_i\}$.

In what follows, let Y be a Banach space and let $C \subset Y$ be a closed convex cone. We use $\le_C$ to denote the binary relation in Y defined by C: for $y_1, y_2 \in Y$, $y_1 \le_C y_2 \Leftrightarrow y_2 - y_1 \in C$; clearly it is a partial order if and only if C is pointed (i.e., $C \cap -C = \{0\}$). In the case when the interior $\mathrm{int}(C)$ of C is nonempty, we define $y_1 <_C y_2$ as $y_2 - y_1 \in \mathrm{int}(C)$. Let C^+ denote the dual cone of C, that is,

$$C^+ = \{c^* \in Y^* : \langle c^*, c\rangle \ge 0 \ \text{ for all } c \in C\}.$$

We denote by C^{+i} the set of all strictly positive continuous linear functionals, that is,

$$C^{+i} = \{c^* \in Y^* : \ \langle c^*, c\rangle > 0 \ \text{ for all } c \in C \setminus \{0\}\}.$$

Recall that C has a base if there exists a convex subset Θ of C such that

$$(*) \qquad C = \{t\theta : \ t \in \mathbb{R}_+ \text{ and } \theta \in \Theta\} \ \text{ and } \ 0 \notin \mathrm{cl}(\Theta),$$

where $\mathrm{cl}(\cdot)$ denotes the closure. We say that C has a bounded (resp., weakly compact) base if it has a base which is bounded (resp., weakly compact). It is known and easy to verify that $C^{+i} \ne \emptyset$ if and only if C has a base.

Now we provide some fundamental notions in vector optimization (cf. [18, 22]).

DEFINITION 2.3. *For $A \subset Y$ and $a \in A$, we say that a is a Pareto (resp., weak-Pareto) efficient point of A if there exists no $y \in A \setminus \{a\}$ (resp., $y \in A$) such that $y \le_C a$ (resp., $y <_C a$). The set of all Pareto (resp., weak-Pareto) efficient points of A is denoted by $\mathrm{E}(A, C)$ (resp., $\mathrm{WE}(A, C)$).*

It is known and easy to verify that

$$a \in \mathrm{E}(A, C) \Leftrightarrow (a - C) \cap A = \{a\} \quad \text{and} \quad a \in \mathrm{WE}(A, C) \Leftrightarrow (a - \mathrm{int}(C)) \cap A = \emptyset.$$

An aim of this paper is to consider the multiobjective optimization problem (1.3) under the assumption that the objective function F is a piecewise linear multifunction between X and Y and that the constraint set Γ is a polyhedron in X. For this aim, we need the following notions.

DEFINITION 2.4. *A point $\bar{x} \in \Gamma$ is said to be a Pareto solution (resp., weak-Pareto solution) of* (1.3) *if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in E(F(\Gamma), C)$ (resp., $\bar{y} \in \mathrm{WE}(F(\Gamma), C)$); in this case, we say that $\bar{y}$ is a Pareto optimal value (resp., weak-Pareto optimal value) of* (1.3). *Let S, S_w, V, and V_w denote, respectively, the set of all Pareto solutions of* (1.3), *the set of all weak-Pareto solutions of* (1.3), *the set of all Pareto optimal values of* (1.3), *and the set of all weak-Pareto optimal values.*

It is clear that

$$S = \Gamma \cap F^{-1}(V) \quad \text{and} \quad S_w = \Gamma \cap F^{-1}(V_w).$$

Arrow, Barankin, and Blackwell, in their pioneering paper [1], established structure theorems on weak-Pareto solution sets, Pareto solution sets, and Pareto optimal value sets of linear multiobjective optimization problems in Euclidean spaces with the ordering cone being polyhedral. These structure theorems have been extended by Zheng and Yang [40] and Zheng [36] to the piecewise linear and multivalued case. In particular, the following two lemmas proved in [36, Theorems 3.1 and 3.2] will be useful for our analysis. For $y^* \in Y^*$, let

$$\lambda_{y^*} := \inf \big\{\langle y^*, y\rangle : \ y \in F(\Gamma)\big\} \ \text{ and } \ L(y^*) := \big\{y \in F(\Gamma) : \ \langle y^*, y\rangle = \lambda_{y^*}\big\}. \tag{2.3}$$

LEMMA 2.6. *Let F be a piecewise linear multifunction from X to Y and let $\Gamma \subset X$ be a polyhedron. Suppose that $F(\Gamma)+C$ is convex. Then the following statements hold:*

(i) *When the ordering cone C has a nonempty interior, there exists $y_i^* \in C^+$ with $\|y_i^*\| = 1$ $(1 \le i \le q)$ such that*

$$V_w = \bigcup_{i=1}^{q} L(y_i^*) \tag{2.4}$$

(and so V_w is the union of finitely many polyhedra in Y);

(ii) *when C has a weakly compact base, there exists $y_i^* \in C^{+i}$ with $\|y_i^*\| = 1$ $(1 \le i \le q)$ such that*

$$V = \bigcup_{i=1}^{q} L(y_i^*) \tag{2.5}$$

(and so V is the union of finitely many polyhedra in Y).

Note (see [36, Example 3.1]) that Lemma 2.6 is not valid if the convexity assumption on $F(\Gamma) + C$ is dropped. However, under the assumption that the ordering cone C is polyhedral, we do have the following (see [36, Theorems 3.3 and 3.4]).

LEMMA 2.7. *Let F be a piecewise linear multifunction from X to Y and let $\Gamma \subset X$ be a polyhedron. Suppose that C is polyhedral and has a nonempty interior. Then V_w is the union of finitely many polyhedra in Y.*

Remark. In contrast, under the assumption on Lemma 2.7, the Pareto optimal value set V is not necessarily the union of finitely many polyhedra in Y (see [36, Example 3.2] for the details).

3. Global metric subregularity for piecewise linear multifucntions. Even though the global metric subregularity is more useful in application and is more interesting in theory, the existing works on the metric subregularity only deal with the local ones except Hoffman's work on the error bound of linear inequalities. In this section, we will consider the global metric subregularity and bounded metric subregularity for piecewise linear (not necessarily convex) multifunctions.

As usual, $\mathbb{N}$ denotes the set of all natural numbers. For $n \in \mathbb{N}$, let

$$\overline{1,n} := \{m \in \mathbb{N} : \ 1 \le m \le n\}.$$

Let $\mathcal{P}$ be a polyhedron in X and $f : \mathcal{P} \to \mathbb{R}$ be a function. Recall that f is piecewise linear if there exist polyhedra $P_1, \dots, P_m$ in X, $\{x_1^*, \dots, x_m^*\} \subset X^*$, and $\{c_1, \dots, c_m\} \subset \mathbb{R}$ such that

$$\mathcal{P} = \bigcup_{i=1}^{m} P_i \ \text{ and } \ f(x) = \langle x_i^*, x\rangle + c_i \quad \forall x \in P_i \text{ and } i \in \overline{1,m}. \tag{3.1}$$

Clearly, (3.1) implies that

$$\langle x_i^*, x\rangle + c_i = \langle x_j^*, x\rangle + c_j \quad \forall x \in P_i \cap P_j.$$

It is easy to verify that if f_1, f_2 are piecewise linear then $f_1 \pm f_2$, $\min\{f_1, f_2\}$, and $\max\{f_1, f_2\}$ are piecewise linear. Let

$$\ker(f) := \{x \in \mathcal{P} :\ f(x) = 0\}. \tag{3.2}$$

It is well known and interesting that if $f_1, f_2 : X \to \mathbb{R}$ are linear then

$$\ker(f_1) \subset \ker(f_2) \Longrightarrow f_2 = \kappa f_1 \text{ for some } \kappa \in \mathbb{R}. \tag{3.3}$$

Of course, with the linearity being replaced by the piecewise linearity, (3.3) is not necessarily true. Nevertheless, for nonnegative piecewise linear functions, we can establish a result somewhat similar to (3.3) (see Theorem 3.1 and Corollary 3.1). To do this, we need several notations. In what follows, let $\mathcal{P}$ be a polyhedron in X and $\phi, \psi : \mathcal{P} \to \mathbb{R}_+$ be piecewise linear functions, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. Then, there exist polyhedra P_i in X, $x_i^*, y_i^* \in X^*$, and $c_i, d_i \in \mathbb{R}$ ($i \in \overline{1,m}$) such that

(3.4)

$$\mathcal{P} = \bigcup_{i=1}^{m} P_i,\ \phi(x) = \langle x_i^*, x\rangle + c_i \text{ and } \psi(x) = \langle y_i^*, x\rangle + d_i \ \ \forall x \in P_i \text{ and } i \in \overline{1,m}.$$

(Indeed, since ϕ and ψ are piecewise linear, we find from the definition polyhedra $Q_j, Q_k' \subset X$ and $(x_j^*, c_j), (y_k^*, d_k) \in X^* \times \mathbb{R}$ ($j \in \overline{1,m_1},\ k \in \overline{1,m_2}$) such that $\mathcal{P} = \bigcup_{j=1}^{m_1} Q_j = \bigcup_{k=1}^{m_2} Q_k'$,

$$\phi(x) = \langle x_j^*, x\rangle + c_j \text{ for } x \in Q_j \ \text{ and } \ \psi(x) = \langle y_k^*, x\rangle + d_k \text{ for } x \in Q_k'.$$

Defining $P_{jk} := Q_j \cap Q_k'$ are polyhedra for $j \in \overline{1,m_1}$ and $k \in \overline{1,m_2}$, we have

$$P = \bigcup_{1\le j\le m_1, 1\le k\le m_2} P_{jk},\ \phi(x) = \langle x_j^*, x\rangle + c_j \text{ and } \psi(x) = \langle y_k^*, x\rangle + d_k \text{ for } x \in P_{jk}.$$

This verifies (3.4)). Let $1 \le i \le m$. By Lemma 2.1, there exist closed subspaces X_i, X_i' of X and a polyhedron P_i' in X_i' such that X_i' is finite dimensional,

$$X = X_i \oplus X_i', \quad \text{and} \quad P_i = X_i + P_i'. \tag{3.5}$$

By [30, Theorem 19.1], the polyhedron P_i' in the finite dimensional space X_i' admits the following representation: there exist $p_i, q_i \in \mathbb{N}$, $a_{ij}, b_{ik} \in X_i'$ ($j \in \overline{1,p_i},\ k \in \overline{1,q_i}$) such that

$$P_i' = \left\{ \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik} :\ \sum_{j=1}^{p_i} \lambda_j = 1,\ \lambda_j, t_k \in \mathbb{R}_+ \right\}. \tag{3.6}$$

The family $\{X_i, X_i', P_i', x_i^*, y_i^*, a_{ij}, b_{ik} :\ i \in \overline{1,m}, j \in \overline{1,p_i}, k \in \overline{1,q_i}\}$ is called a frame for (ϕ, ψ) if (3.4), (3.5), and (3.6) are satisfied. Note from (3.5) and (3.6) that $x \in P_i$ if and only if there exists

$$(x_i, \lambda_j, t_k) \in X_i \times \mathbb{R}_+ \times \mathbb{R}_+ \ (j \in \overline{1,p_i},\ k \in \overline{1,q_i}) \ \text{ with } \ \sum_{j=1}^{p_i} \lambda_j = 1 \tag{3.7}$$

such that

$$x = x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}; \tag{3.8}$$

in this case we say that $\{(x_i, \lambda_j, t_k) : \ j \in \overline{1, p_i}, \ k \in \overline{1, q_i}\}$ is a framed representation of $x \in P_i$. For any $i \in \overline{1, m}$ and any (x_i, λ_j, t_k) satisfying (3.7), we note for later reference that

$$\phi\left(x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) = \langle x_i^*, x_i\rangle + \sum_{j=1}^{p_i} \lambda_j \phi(a_{ij}) + \sum_{k=1}^{q_i} t_k \langle x_i^*, b_{ik}\rangle \tag{3.9}$$

and

$$\psi\left(x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) = \langle y_i^*, x_i\rangle + \sum_{j=1}^{p_i} \lambda_j \psi(a_{ij}) + \sum_{k=1}^{q_i} t_k \langle y_i^*, b_{ik}\rangle \tag{3.10}$$

(by (3.4)).

For simplicity of presentation and the proofs of Lemma 3.1 and Theorem 3.1, it would be convenient to introduce several notations for several index subsets: Let

$$I_0 := \{i \in \overline{1, m} : \ P_i \cap \ker(\phi) = \emptyset\}, \ \ I_1 := \overline{1, m} \setminus I_0. \tag{3.11}$$

For $i \in \overline{1, m}$, let

$$\begin{aligned} J_i(\phi) &:= \{j \in \overline{1, p_i} : \ \phi(a_{ij}) = 0\}, \quad J_i^+(\phi) := \overline{1, p_i} \setminus J_i(\phi), \\ J_i(\psi) &:= \{j \in \overline{1, p_i} : \ \psi(a_{ij}) = 0\}, \quad J_i^+(\psi) := \overline{1, p_i} \setminus J_i(\psi), \end{aligned} \tag{3.12}$$

$$\begin{aligned} K_i(\phi) &:= \{k \in \overline{1, q_i} : \ \langle x_i^*, b_{ik}\rangle = 0\}, \quad K_i^+(\phi) := \overline{1, q_i} \setminus K_i(\phi), \\ K_i(\psi) &:= \{k \in \overline{1, q_i} : \ \langle y_i^*, b_{ik}\rangle = 0\}, \quad K_i^+(\psi) := \overline{1, q_i} \setminus K_i(\psi). \end{aligned} \tag{3.13}$$

Finally, we define $\kappa_\psi(\phi)$ by

$$\kappa_\psi(\phi) := \min\left\{\phi(a_{ij}) : \ i \in I_0 \text{ and } j \in \overline{1, p_i}\right\}, \tag{3.14}$$

where $\kappa_\psi(\phi)$ is understood as $+\infty$ if $I_0 = \emptyset$.

LEMMA 3.1. *Let $\mathcal{P}$ be a polyhedron in X and ϕ, ψ be nonnegative-valued piecewise linear functions on $\mathcal{P}$ with the associate notations $P_i, x_i^*, c_i, y_i^*, d_i, X_i, x_i', P_i'$, a_{ij}, b_{ij} as explained in (3.4)–(3.10). Let the notations $I_0, J_i(\phi), J_i^+(\phi), J_i(\psi), J_i^+(\psi)$, $K_i(\phi), K_i^+(\phi), K_i(\psi), K_i^+(\psi)$, and $\kappa_\psi(\phi)$ be explained as in (3.11)–(3.14). Then the following statements hold for each $i \in \overline{1, m}$ and each $k \in \overline{1, q_i}$:*

(i) $X_i \subset \ker(x_i^*) \cap \ker(y_i^*)$;

(ii) $0 \le \langle x_i^*, b_{ik}\rangle$ *and* $0 \le \langle y_i^*, b_{ik}\rangle$;

(iii) $P_i \cap \ker(\phi)$
$= X_i + \{\sum_{j\in J_i(\phi)} \lambda_j a_{ij} + \sum_{k \in K_i(\phi)} t_k b_{ik} : \ \sum_{j \in J_i(\phi)} \lambda_j = 1, \ \lambda_j, t_k \in \mathbb{R}_+\}$;

(iv) $P_i \cap \ker(\psi)$
$= X_i + \{\sum_{j\in J_i(\psi)} \lambda_j a_{ij} + \sum_{k \in K_i(\psi)} t_k b_{ik} : \ \sum_{j \in J_i(\psi)} \lambda_j = 1, \ \lambda_j, t_k \in \mathbb{R}_+\}$;

(v) $0 < \kappa_\psi(\phi) \le \inf\{\phi(x) : \ x \in \bigcup_{i \in I_0} P_i\}$.

Proof. Let $1 \leq i \leq m$. Since ϕ is nonnegative, (3.9) implies that $\langle x_i^*, x_i \rangle = 0$ for all $x_i \in X_i$ (because X_i is a subspace) and that $\langle x_i^*, b_{ik} \rangle \geq 0$ for all $k \in \overline{1, q_i}$ (by considering $t_k \to \infty$ in (3.9)). Thus (i) and (ii) are seen to hold as the corresponding assertions regarding ψ can be shown similarly.

To prove (iii), let A denote the set on the right-hand side of (iii), and we need only show that

$$P_i \cap \ker(\phi) \subset A, \tag{3.15}$$

where $\ker(\cdot)$ is as in (3.2). To check this, let $x \in P_i \cap \ker(\phi)$. Then there exists (x_i, λ_j, t_k) satisfying (3.7) such that $x = x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}$, and it follows from (3.9) that

$$0 = \langle x_i^*, x_i \rangle + \sum_{j=1}^{p_i} \lambda_j \phi(a_{ij}) + \sum_{k=1}^{q_i} t_k \langle x_i^*, b_{ik} \rangle. \tag{3.16}$$

Noting (by (i), (ii), and the nonnegativity of ϕ) that each term on the right-hand side of (3.16) is nonnegative, this implies that

$$\lambda_j \phi(a_{ij}) = 0 \text{ and } t_k \langle x_i^*, b_{ik} \rangle = 0 \quad \forall (j, k) \in \overline{1, p_i} \times \overline{1, q_i},$$

that is, $\lambda_j = 0$ for all $j \in \overline{1, p_i} \setminus J_i(\phi)$ and $t_k = 0$ for all $k \in \overline{1, q_i} \setminus K_i(\phi)$. Hence $x \in A$. This shows that (3.15) holds. Similarly, one can show that (iv) also holds. Since the second inequality of (v) is immediate from (i), (ii), (3.9), and the definition of $\kappa_\psi(\phi)$ in (3.14), it remains to show the first inequality of (v). Let $i \in I_0$ and $j \in \overline{1, p_i}$. We need only to show that $0 < \phi(a_{ij})$, but this is evident as $P_i \cap \ker(\phi) = \emptyset$ (because of (3.11)) and $a_{ij} \in P_i' \subset P_i$. This completes the proof. $\square$

Lemma 3.1 is a useful tool for the proofs of the main results. We also need the following lemma.

LEMMA 3.2. *Continuing the notations for $\mathcal{P}, \phi, \psi$, etc., as in Lemma* 3.1, *suppose that*

$$\ker(\phi) \subset \ker(\psi). \tag{3.17}$$

Let $i \in I_1$, where I_1 is as (3.11). *Then the following statements hold:*

(i) *For any (x_i, λ_j, t_k) satisfying* (3.7), *we have*

$$\phi\left(x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) = \sum_{j \in J_i^+(\phi)} \lambda_j \phi(a_{ij}) + \sum_{k \in K_i^+(\phi)} t_k \langle x_i^*, b_{ik} \rangle \tag{3.18}$$

and

$$\psi\left(x_i + \sum_{j=1}^{p_i} \lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) = \sum_{j \in J_i^+(\phi)} \lambda_j \psi(a_{ij}) + \sum_{k \in K_i^+(\phi)} t_k \langle y_i^*, b_{ik} \rangle, \tag{3.19}$$

where $J_i^+(\phi)$ and $K_i^+(\phi)$ are as in (3.12) *and* (3.13), *respectively;*

(ii) *there exists $\eta_i \in (0, +\infty)$ such that $\eta_i \psi(x) \leq \phi(x)$ for all $x \in P_i$.*

Proof. By Lemma 3.1(i), (3.9), (3.12), and (3.13), (3.18) is clear. For (3.19), the argument is similar provided that the pair of inclusions $J_i^+(\psi) \subset J_i^+(\phi)$ and $K_i^+(\psi) \subset K_i^+(\phi)$ hold, which is equivalent to the following inclusions holding:

$$J_i(\phi) \subset J_i(\psi) \text{ and } K_i(\phi) \subset K_i(\psi), \tag{3.20}$$

where $J_i(\phi)$ and $K_i(\phi)$ are as in (3.13). Thus, for (i), it suffices to show (3.20). The first inclusion of (3.20) is immediate from (3.17) and, for the second inclusion, we suppose to the contrary that there exists $k_0 \in K_i(\phi) \setminus K_i(\psi)$, that is,

$$\langle x_i^*, b_{ik_0}\rangle = 0 \text{ and } \langle y_i^*, b_{ik_0}\rangle > 0. \tag{3.21}$$

Since $i \in I_1$, there exists $u_i \in P_i$ such that $\phi(u_i) = 0$, and so $\psi(u_i) = 0$ by (3.17). Noting (by (3.5) and (3.6)) that $u_i + b_{ik_0} \in P_i$, it follows from (3.9) and (3.10) that

$$\phi(u_i + b_{ik_0}) = \phi(u_i) + \langle x_i^*, b_{ik_0}\rangle = 0 \text{ and } \psi(u_i + b_{ik_0}) = \psi(u_i) + \langle y_i^*, b_{ik_0}\rangle > 0.$$

This contradicts (3.17) and so (i) is established. For (ii), let

$$\tau_1(i) := \min\big\{ \min_{j\in J_i^+(\phi)} \phi(a_{ij}),\ \min_{k\in K_i^+(\phi)} \langle x_i^*, b_{ik}\rangle\big\}$$

and

$$\tau_2(i) := \max\big\{ \max_{j\in J_i^+(\phi)} \psi(a_{ij}),\ \max_{k\in K_i^+(\phi)} \langle y_i^*, b_{ik}\rangle\big\}.$$

Since (ii) holds trivially when $\psi(x) = 0$ for all $x \in P_i$, we assume that $\psi(x_0) > 0$ for some $x_0 \in P_i$. Then, by the definitions of $J_i^+(\phi)$ and $K_i^+(\phi)$, one has

$$0 < \tau_1(i),\ \ 0 < \tau_2(i),\ \ \frac{\tau_1(i)\psi(a_{ij})}{\tau_2(i)} \le \phi(a_{ij}),\ \text{ and } \frac{\tau_1(i)\langle y_i^*, b_{ik}\rangle}{\tau_2(i)} \le \langle x_i^*, b_{ik}\rangle$$

for all $(i,k) \in J_i^+(\phi) \times K_i^+(\phi)$. It follows from (i) that (ii) holds with $\eta_i = \frac{\tau_1(i)}{\tau_2(i)}$. The proof is complete. ◻

Now we are ready to prove a result corresponding to (3.3) for two nonnegative piecewise linear functions. This result not only is of independent interest but also plays a key role in the proof of our main results.

THEOREM 3.1. *Let $\mathcal{P}$ be a polyhedron in X and consider nonnegative-valued piecewise linear functions ϕ, ψ on $\mathcal{P}$ with the associate notations $P_i, x_i^*, c_i, y_i^*, d_i, X_i, x_i', P_i'$, a_{ij}, b_{ij} as explained in* (3.4)–(3.10). *Let notations $I_1, I_0, J_i(\phi), J_i^+(\phi), J_i(\psi), J_i^+(\psi)$, $K_i(\phi), K_i^+(\phi), K_i(\psi), K_i^+(\psi)$, and $\kappa_\psi(\phi)$ be explained as in* (3.11)–(3.14). *Suppose that* (3.17) *holds. Then the following statements hold:*

(i) *Suppose that* $\sup_{x\in\mathcal{P}} \psi(x) < +\infty$. *Then there exists $\eta > 0$ such that*

$$\eta\psi(x) \le \phi(x) \quad \forall x \in \mathcal{P}; \tag{3.22}$$

(ii) *suppose that* $\sup\{\psi(x) :\ x \in \mathcal{P}\} = +\infty$. *Then*

$$\kappa_\psi(\phi) \le \liminf_{\psi(x)\to\infty} \phi(x) \tag{3.23}$$

and there exists some $\eta > 0$ such that

$$\eta\psi(x) \le \phi(x) \quad \forall x \in \mathcal{P} \text{ with } \phi(x) < \liminf_{\psi(y)\to\infty} \phi(y); \tag{3.24}$$

(iii) *suppose that* $\sup\{\psi(x) :\ x \in \mathcal{P}\} = +\infty$ *and that* $\liminf_{\psi(x)\to\infty} \phi(x) < +\infty$. *Then there exists a sequence $\{x_n\}$ in $\mathcal{P}$ such that $\psi(x_n) \to \infty$ and* $\phi(x_n) = \liminf_{\psi(x)\to\infty} \phi(x)$ *for all $n \in \mathbb{N}$ (in particular, the strict inequality assumption in* (3.24) *cannot be replaced by the nonstrict one).*

Proof. For each $i \in I_1$ (see (3.11)), take $\eta_i > 0$ having the property stated in Lemma 3.2(ii):

$$\eta_i\psi(x) \le \phi(x) \quad \forall x \in P_i,\ i \in I_1.$$

Let $I_0^\infty := \{i \in I_0 :\ \sup_{x\in P_i}\psi(x) = +\infty\}$, where I_0 is as in (3.11). For each $i \in I_0 \setminus I_0^\infty$, we define $\eta_i := \frac{\kappa_\psi(\phi)}{\sup_{x\in P_i}\psi(x)}$. Then, by Lemma 3.1(v), we have

$$\eta_i\psi(x) \le \phi(x) \quad \forall x \in P_i,\ i \in I_0 \setminus I_0^\infty.$$

Thus, noting that $\overline{1,m} = I_0 \cup I_1$ and letting $\bar\eta := \min\{\eta_i :\ i \in \overline{1,m} \setminus I_0^\infty\}$, we have $\bar\eta > 0$ and

$$\bar\eta\psi(x) \le \phi(x) \quad \forall x \in \bigcup_{i\in\overline{1,m}\setminus I_0^\infty} P_i. \tag{3.25}$$

Since $\mathcal{P} = \bigcup_{i=1}^m P_i$, (i) follows immediately from (3.25) (because $I_0^\infty = \emptyset$ if $\sup_{x\in\mathcal{P}}\psi(x) < +\infty$). To prove (ii) and (iii), suppose that $\sup_{x\in\mathcal{P}}\psi(x) = +\infty$. If $I_0^\infty = \emptyset$, (3.25) also implies that $\liminf_{\psi(x)\to\infty}\phi(x) = +\infty$ and (ii) holds. So we can assume henceforth that $I_0^\infty \neq \emptyset$. By (3.25), we have

$$\liminf_{\psi(x)\to\infty}\phi(x) = \min\Big\{\liminf_{x\in P_i,\psi(x)\to\infty}\phi(x) :\ i \in I_0^\infty\Big\}. \tag{3.26}$$

This and Lemma 3.1(v) imply that (3.23) holds. Let $i \in I_0^\infty$, $j_i \in \overline{1,p_i}$ be such that $\phi(a_{ij_i}) = \min_{1\le j\le p_i}\phi(a_{ij})$ and let

$$K_i^+(\phi,\psi) := \{k \in K_i^+(\psi) :\ \langle x_i^*, b_{ik}\rangle > 0\},$$

where $K_i^+(\psi)$ is as in (3.13). By (3.9), (3.10), (3.13), (i) and (v) of Lemma 3.1, and the definition of $K_i^+(\psi)$, we have

$$\phi\left(x_i + \sum_{j=1}^{p_i}\lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) \ge \phi(a_{ij_i}) + \sum_{k\in K_i^+(\phi,\psi)} t_k\langle x_i^*, b_{ik}\rangle \tag{3.27}$$

and

$$\psi\left(x_i + \sum_{j=1}^{p_i}\lambda_j a_{ij} + \sum_{k=1}^{q_i} t_k b_{ik}\right) = \sum_{j=1}^{p_i}\lambda_j\psi(a_{ij}) + \sum_{k\in K_i^+(\psi)} t_k\langle y_i^*, b_{ik}\rangle \tag{3.28}$$

for all (x_i, λ_j, t_k) satisfying (3.7). Note in particular that a_{ij_i} is a minimizer of ϕ on P_i. Let

$$\mathcal{K}_0^\infty := \big\{i \in I_0^\infty :\ K_i^+(\psi) = K_i^+(\phi,\psi)\big\}.$$

Then, by (3.27) and (3.28), we have

$$\eta_i\psi(x) \le \phi(x) \quad \forall x \in P_i,\ i \in \mathcal{K}_0^\infty, \tag{3.29}$$

where $\eta_i := \frac{\min\{\phi(a_{ij_i}),\ \min_{k\in K_i^+(\psi)}\langle x_i^*, b_{ik}\rangle\}}{\max\{\max_{1\le j\le p_i}\psi(a_{ij}),\ \max_{k\in K_i^+(\psi)}\langle y_i^*, b_{ik}\rangle\}}$. This and (3.26) imply that

$$\liminf_{\psi(x)\to\infty}\phi(x) = \inf\Big\{\liminf_{x\in P_i,\psi(x)\to\infty}\phi(x) :\ i \in I_0^\infty \setminus \mathcal{K}_0^\infty\Big\}. \tag{3.30}$$

Let i be an arbitrary element in $I_0^\infty \setminus \mathcal{K}_0^\infty$ and take $k_i \in K_i^+(\psi) \setminus K_i^+(\phi, \psi)$. Then

$$\langle x_i^*, b_{ik_i}\rangle = 0 \ \text{ and } \ \langle y_i^*, b_{ik_i}\rangle > 0.$$

It follows from (3.9) and (3.10) that

$$\phi(a_{ij_i} + nb_{ik_i}) = \phi(a_{ij_i}) \quad \forall n \in \mathbb{N} \tag{3.31}$$

and

$$\psi(a_{ij_i} + nb_{ik_i}) = \psi(a_{ij_i}) + n\langle y_i^*, b_{ik_i}\rangle \to \infty \ \text{ as } n \to \infty. \tag{3.32}$$

Thus, by (3.30) and (3.27), we have

$$\liminf_{\psi(x)\to\infty} \phi(x) \leq \lim_{n\to\infty} \phi(a_{ij_i} + nb_{ik_i}) = \phi(a_{ij_i}) = \min_{x\in P_i} \phi(x) \quad \forall i \in I_0^\infty \setminus \mathcal{K}_0^\infty. \tag{3.33}$$

This, together with (3.25) and (3.29), implies that (3.24) holds with $\eta = \min\{\bar{\eta}, \min_{i\in\mathcal{K}_0^\infty} \eta_i\}$. It remains to show (iii). To do this, further suppose that $\liminf_{\psi(x)\to\infty} \phi(x) < +\infty$. Thus, as shown above, $I_0^\infty \setminus \mathcal{K}_0^\infty$ must be nonempty. By (3.30) and (3.33), there exists $i \in I_0^\infty \setminus \mathcal{K}_0^\infty$ such that

$$\liminf_{\psi(x)\to\infty} \phi(x) = \phi(a_{ij_i}),$$

and hence one can take $k_i \in K_i^+(\psi) \setminus K_i^+(\phi, \psi)$ satisfying (3.31) and (3.32). This shows that (iii) holds with $x_n = a_{ij_i} + nb_{ik_i}$ for each $n \in \mathbb{N}$. The proof is completed. ◻

The following corollary is immediate from Theorem 3.1.

Corollary 3.1. *Let $\mathcal{P}, \phi, \psi$ be as in Lemma* 3.1 *and suppose that* (3.17) *holds. Then there exist $\eta, \kappa \in (0, +\infty)$ such that*

$$\eta\psi(x) \leq \phi(x) \quad \forall x \in \mathcal{P} \text{ with } \phi(x) \leq \kappa.$$

Moreover there exists $\eta > 0$ such that

$$\eta\psi(x) \leq \phi(x) \quad \forall x \in \mathcal{P}$$

if and only if $\lim_{\psi(x)\to\infty} \phi(x) = \infty$.

This corollary together with the following two lemmas will enable us to establish results on metric subregularity for piecewise linear multifunctions.

Lemma 3.3. *Let $G : X \rightrightarrows Y$ be a linear multifunction and let $b \in Y$. Then there exist $x_i^* \in X^*$ and $t_i \in \mathbb{R}$ $(1 \leq i \leq m)$ such that*

$$\eta_1 \max_{1\leq i\leq m} (\langle x_i^*, x\rangle + t_i) \leq d(b, G(x)) \leq \eta_2 \max_{1\leq i\leq m} (\langle x_i^*, x\rangle + t_i) \ \ \forall x \in \mathrm{dom}(G), \tag{3.34}$$

where $\eta_1, \eta_2 \in (0, +\infty)$ are constants.

Proof. Let $(x_i^*, y_i^*) \in X^* \times Y^*$ and $c_i \in \mathbb{R}$ $(1 \leq i \leq m)$ be such that

$$\mathrm{Gr}(G) = \{(x, y) \in X \times Y : \ \langle x_i^*, x\rangle + \langle y_i^*, y\rangle \leq c_i, \ 1 \leq i \leq m\}.$$

Then,

$$G(x) = \{y \in Y : \ \langle y_i^*, y\rangle \leq c_i - \langle x_i^*, x\rangle, \ 1 \leq i \leq m\} \ \ \forall x \in X.$$

By Lemma 2.5, there exists $\eta_2 \in (0, +\infty)$, only dependent on $\{y_i^*: 1 \le i \le m\}$, such that

$$d(b, G(x)) \le \eta_2 \max_{i \in I}[\langle y_i^*, b\rangle - c_i + \langle x_i^*, x\rangle]_+ \;\; \forall x \in \mathrm{dom}(G), \tag{3.35}$$

where $[\alpha]_+ := \max\{\alpha, 0\}$. Let $I := \{1 \le i \le m: \; y_i^* \ne 0\}$. If $I = \emptyset$, then it is easy to verify that $G(x) = Y$ for all $x \in \mathrm{dom}(G)$, and so (3.34) trivially holds with $x_i^* = 0$, $t_i = 0$, and $\eta_1 = \eta_2 = 1$. Now we assume that $I \ne \emptyset$. Let $x \in \mathrm{dom}(G)$ and

$$L_i(x) := \{y \in Y: \; \langle y_i^*, y\rangle \le c_i - \langle x_i^*, x\rangle\}.$$

Then $G(x) = \bigcap_{i \in I} L_i(x)$. Letting $\eta_1 := \min_{i \in I} \frac{1}{\|y_i^*\|}$ and noting that, by an elementary functional analysis argument,

$$d(b, L_i(x)) = \frac{1}{\|y_i^*\|}[\langle y_i^*, b\rangle - c_i + \langle x_i^*, x\rangle]_+,$$

it follows that

$$\eta_1 \max_{i \in I}[\langle y_i^*, b\rangle + \langle x_i^*, x\rangle - c_i]_+ \le \max_{i \in I} d(b, L_i(x)) \le d(b, G(x)).$$

This and (3.35) imply that (3.34) holds with $t_i = \langle y_i^*, b\rangle - c_i$. The proof is completed. ∎

Similarly to the second part of the proof of Lemma 3.3, we can prove the following Lemma 3.4.

LEMMA 3.4. *Let P be a polyhedron of X. Then, there exist $x_i^* \in X^*$ and $t_i \in \mathbb{R}$ $(1 \le i \le m)$ such that*

$$\eta_1 \max_{1 \le i \le m}(\langle x_i^*, x\rangle + t_i) \le d(x, P) \le \eta_2 \max_{1 \le i \le m}(\langle x_i^*, x\rangle + t_i) \;\; \forall x \in X,$$

where $\eta_1, \eta_2 \in (0, +\infty)$ are constants.

For a linear multifunction G between Banach spaces X and Y and a polyhedron P of X, the functions $x \mapsto d(b, G(x))$ and $x \mapsto d(x, P)$ are not necessarily piecewise linear. Nevertheless, Lemmas 3.3 and 3.4 imply that each of these functions is bounded above and below by multiples of a convex nonnegative piecewise linear function.

It is worth noting that x_i^* and t_i in Lemma 3.3 are dependent on b while they are independent on x in both Lemmas 3.3 and 3.4.

Now we are ready to provide a characterization for the global metric subregualrity.

THEOREM 3.2. *Let $F: X \rightrightarrows Y$ be a piecewise linear multifunction. Let $\Gamma \subset X$ and $E \subset Y$ be unions of finitely many polyhedra. Then F is globally metrically subregular for (Γ, E) if and only if*

$$\lim_{d(x, F^{-1}(E) \cap \Gamma) \to \infty} (d(E, F(x)) + d(x, \Gamma)) = \infty. \tag{3.36}$$

Proof. The necessity part is trivial. To prove the sufficiency part, suppose that (3.36) holds. Since F is piecewise linear, and Γ, E are unions of finitely many polyhedra, there exist multifunctions G_i, $P_j \subset X$, and $E_k \subset Y$ $(i \in \overline{1, m_1}, \; j \in \overline{1, m_2}$, $k \in \overline{1, m_3})$ such that $\mathrm{Gr}(G_i), P_j, E_k$ are polyhedra and

$$\mathrm{Gr}(F) = \bigcup_{i=1}^{m_1} \mathrm{Gr}(G_i), \;\; \Gamma = \bigcup_{j=1}^{m_2} P_j, \;\; E = \bigcup_{k=1}^{m_3} E_k. \tag{3.37}$$

Let $\mathcal{I} := \{(i,j,k) \in \overline{1,m_1} \times \overline{1,m_2} \times \overline{1,m_3} :\ G_i^{-1}(E_k) \cap P_j \neq \emptyset\}$. Clearly one has

$$F^{-1}(E) \cap \Gamma = \bigcup_{(i,j,k)\in\mathcal{I}} G_i^{-1}(E_k) \cap P_j, \tag{3.38}$$

$$d(x, F^{-1}(E) \cap \Gamma) = \min\left\{d(x, G_i^{-1}(E_k) \cap P_j) :\ (i,j,k) \in \mathcal{I}\right\}, \tag{3.39}$$

and

$$d(E, F(x)) + d(x,\Gamma) = \min\left\{d(E_k, G_i(x)) + d(x,\Gamma) :\ i \in \overline{1,m_1},\ k \in \overline{1,m_3}\right\} \tag{3.40}$$

for all $x \in X$. Let $(i,j,k) \in \mathcal{I}$. Then $G_i^{-1}(E_k) \cap P_j = (\mathrm{Gr}(G_i) \cap (P_j \times E_k))_X$ is a polyhedron in X (see Lemma 2.2(i)) and the function G_{ik}, defined by $G_{ik}(\cdot) = G_i(\cdot) - E_k$, is a linear multifunction as $\mathrm{Gr}(G_{ik}) = \mathrm{Gr}(G_i) - \{0\} \times E_k$ is a polyhedron in $X \times Y$ (see Lemma 2.2(iii)). Thus, by Lemmas 3.4 and 3.3, there exist nonnegative and piecewise linear functions ψ_{ijk} on X and ϕ_{ik} on $\mathrm{dom}(G_i)$ together with constants $\eta_{ijk}, \eta'_{ijk}, \eta_{ik}, \eta'_{ik} \in (0,\ +\infty)$ such that

$$\eta_{ijk}\psi_{ijk}(x) \le d(x, G_i^{-1}(E_k) \cap P_j) \le \eta'_{ijk}\psi_{ijk}(x) \quad \forall x \in X \tag{3.41}$$

and

$$\eta_{ik}\phi_{ik}(x) \le d(0, G_{ik}(x)) = d(E_k, G_i(x)) \le \eta'_{ik}\phi_{ik}(x) \quad \forall x \in \mathrm{dom}(G_i). \tag{3.42}$$

Similarly, one can find $\bar\eta, \bar\eta' \in (0,\ +\infty)$ and a nonnegative piecewise linear function $\bar\phi$ on X such that

$$\bar\eta\bar\phi(x) \le d(x,\Gamma) \le \bar\eta'\bar\phi(x) \quad \forall x \in X. \tag{3.43}$$

Let $\psi : X \to \mathbb{R}_+$ be the piecewise linear function defined by

$$\psi(x) = \min\left\{\psi_{ijk}(x) :\ (i,j,k) \in \mathcal{I}\right\} \quad \forall x \in X$$

and let $\eta_1, \eta_2 \in (0,\ +\infty)$ be defined by

$$\eta_1 := \min\{\eta_{ijk} :\ (i,j,k) \in \mathcal{I}\} \ \text{ and } \ \eta'_1 := \max\{\eta'_{ijk} :\ (i,j,k) \in \mathcal{I}\}.$$

Then, by (3.39) and (3.41), we have

$$\eta_1\psi(x) \le d(x, F^{-1}(E) \cap \Gamma) \le \eta'_1\psi(x) \quad \forall x \in X. \tag{3.44}$$

On the other hand, for any $(i,k) \in \overline{1,m_1} \times \overline{1,m_3}$, let $\phi_{ik} : \mathrm{dom}(G_i) \to \mathbb{R}_+$ be the piecewise linear function defined by

$$\tilde\phi_{ik}(x) := \phi_{ik}(x) + \bar\phi(x) \quad \forall x \in \mathrm{dom}(G_i)$$

and let $\tau_{ik} := \min\{\eta_{ik}, \bar\eta\}$ and $\tau'_{ik} := \max\{\eta'_{ik}, \bar\eta'\}$. Then, by (3.42) and (3.43), we also have

$$\tau_{ik}\tilde\phi_{ik}(x) \le d(E_k, G_i(x)) + d(x,\Gamma) \le \tau'_{ik}\tilde\phi_{ik}(x) \quad \forall x \in \mathrm{dom}(G_i). \tag{3.45}$$

By (3.44), (3.38), and (3.45), one has

$$\ker(\psi) = F^{-1}(E) \cap \Gamma \supset G_i^{-1}(E_k) \cap \Gamma = \ker(\tilde\phi_{ik}). \tag{3.46}$$

Moreover, by (3.44), (3.36), (3.40), and (3.45), one has

$$\lim_{\psi(x)\to\infty} \tilde{\phi}_{ik}(x) = \infty.$$

It follows from (3.46) and Corollary 3.1 that there exists $\beta_{ik} \in (0,\ +\infty)$ such that

$$\beta_{ik}\psi(x) \le \tilde{\phi}_{ik}(x) \quad \forall x \in \mathrm{dom}(G_i).$$

Thus, by (3.44) and (3.45), one has

$$\frac{\beta_{ik}\tau_{ik}}{\eta_1'} d(x, F^{-1}(E) \cap \Gamma) \le d(E_k, G_i(x)) + d(x, \Gamma) \quad \forall x \in X.$$

This and (3.40) imply that

$$\eta d(x, F^{-1}(E) \cap \Gamma) \le d(E, F(x)) + d(x, \Gamma) \quad \forall x \in X$$

with $\eta := \min\{\frac{\beta_{ik}\tau_{ik}}{\eta_1'} :\ i \in \overline{1, m_1},\ k \in \overline{1, m_3}\}$. The proof is complete. ∎

Remark. It is clear that (3.36) holds if Γ is bounded. Therefore, under the assumption of Theorem 3.2, F is always globally metrically subregular for (Γ, E) if Γ is bounded.

THEOREM 3.3. *Let F, Γ, and E be as in Theorem* 3.2. *The following statements hold:*

(i) *There exist $\kappa, \eta \in (0,\ +\infty)$ such that*

$$\eta d(x, F^{-1}(E)\cap\Gamma) \le d(E, F(x)) + d(x, \Gamma) \ \ \forall x \in X \text{ with } d(E, F(x)) + d(x, \Gamma) < \kappa;$$

(ii) *for any $r \in (0,\ +\infty)$ there exists $\eta_r \in (0,\ +\infty)$ such that*

$$\eta_r d(x, F^{-1}(E)\cap\Gamma) \le d(E, F(x)) + d(x, \Gamma) \ \ \forall x \in F^{-1}(E)\cap\Gamma + rB_X; \tag{3.47}$$

consequently, F is boundedly metrically subregular for (Γ, E).

Proof. The proof for (i) is the same as that for Theorem 3.2 (except that one uses the second assertion in place of the first assertion in Corollary 3.1). For (ii), let $\eta_r := \min\{\eta, \frac{\kappa}{r}\}$ for all $r \in (0,\ +\infty)$, where η and κ are as in (i). For any $r \in (0,\ +\infty)$, noting that $d(x, F^{-1}(E) \cap \Gamma) \le r$ for all $x \in F^{-1}(E) \cap \Gamma + rB_X$, it is easy to verify that (3.47) holds and so does (ii). ∎

We conclude this section with an interesting application regarding linear regularity for a collection of finitely many closed sets. Let $\Theta_1, \dots, \Theta_m$ be closed subsets of X with their intersection $\Theta := \bigcap_{i=1}^m \Theta_i$ nonempty. Recall that the collection $\{\Theta_1, \dots, \Theta_m\}$ is linearly regular (resp., boundedly linearly regular) if there exists $\eta \in (0,\ +\infty)$ (resp., for any $r \in (0,\ +\infty)$ there exists $\eta \in (0,\ +\infty)$) such that

$$\eta d(x, \Theta) \le \max_{1\le i\le m} d(x, \Theta_i) \ \ \forall x \in X \text{ (resp., } \forall \mathrm{x} \in \mathrm{rB_X}).$$

Linear regularity is a fundamental notion in mathematical programming and approximation theory and has been studied extensively (see [5, 2, 37]). Under the polyhedron assumption, Bauschke and Borwein [4] and Bauschke, Borwein, and Li [5] provided the following result.

THEOREM BBL. *Suppose that $\Theta_1, \dots, \Theta_m$ are polyhedra in Eucildean space $\mathbb{R}^n$ with their intersection Θ nonempty. Then the collection $\{\Theta_1, \dots, \Theta_m\}$ is linearly regular.*

The following result considers the case when each Θ_i is the union of finitely many polyhedra.

THEOREM 3.4. *Let X be a Banach space and Θ_i be the union of finitely many polyhedra in X $(1 \le i \le m)$ such that $\Theta := \bigcap_{i=1}^m \Theta_i$ is nonempty. Then the following statements hold:*

(i) *There exist $\eta, \kappa \in (0, +\infty)$ such that*

$$\eta d(x, \Theta) \le \max_{1\le i\le m} d(x, \Theta_i) \quad \forall x \in X \text{ with } \max_{1\le i\le m} d(x, \Theta_i) < \kappa;$$

(ii) $\{\Theta_1, \ldots, \Theta_m\}$ *is boundedly linearly regular;*

(iii) $\{\Theta_1, \ldots, \Theta_m\}$ *is linearly regular if and only if* $\lim_{d(x,\Theta)\to\infty} \max_{1\le i\le m} d(x, \Theta_i) = \infty$.

Proof. Let $\Gamma = X$ and define $F : X \rightrightarrows X^m$ as $F(x) = (x, \ldots, x)$ for all $x \in X$, where X^m is equipped with the norm $\|(x_1, \ldots, x_m)\| = \max_{1\le i\le m} \|x_i\|$ for all $(x_1, \ldots, x_m) \in X^m$. From Lemma 2.1, without lost of generality, we assume that X is finite dimensional. Then, $\mathrm{Gr}(F)$ is a linear subspace of $X \times X^m$ and so a polyhedron. Note that $\Theta = F^{-1}(\Theta_1 \times \cdots \times \Theta_m)$ and $d(\Theta_1 \times \cdots \times \Theta_m, F(x)) = \max_{1\le i\le m} d(x, \Theta_i)$. Thus, the conclusion follows from Theorems 3.3 and 3.2. ∎

4. Weak sharp minima. In this section, as applications of Theorems 3.2 and 3.3, we consider weak sharp minima for the piecewise linear multiobjective optimization problem (1.3), with F, X, Y, C, and Γ as indicated there (S_w, V_w, S, V are defined in section 2).

THEOREM 4.1. *Suppose that $F(\Gamma) + C$ is convex. Then the following statements hold:*

(i) *When C has a nonempty interior, problem* (1.3) *always has the bounded weak sharp minimum property with respect to weak-Pareto optimal values for* (1.3). *Moreover* (1.3) *has the global weak sharp minimum property with respect to weak-Pareto optimal values for* (1.3) *if and only if*

$$\lim_{d(x,S_w)\to\infty} d(V_w, F(x)) + d(x, \Gamma) = +\infty;$$

(ii) *when C has a weakly compact base,* (1.3) *always has the bounded weak sharp minimum property with respect to Pareto optimal values for* (1.3). *Moreover,* (1.3) *has the global weak sharp minimum property with respect to Pareto optimal values for* (1.3) *if and only if*

$$\lim_{d(x,S)\to\infty} d(V, F(x)) + d(x, \Gamma) = +\infty.$$

Proof. By Lemma 2.6(i), the weak-Pareto optimal value set V_w is the union of finitely many polyhedra in Y. Noting that $S_w = F^{-1}(V_w) \cap \Gamma$, (1.3) has the bounded (resp., global) weak sharp minimum property with respect to weak-Pareto optimal values for (1.3) if and only if F is boundedly (resp., globally) metrically subregular for (V_w, Γ). Hence (i) follows from Theorems 3.3 and 3.2. When C has a weakly compact base, Lemma 2.6(ii) implies that the Pareto optimal value set V is the union of finitely many polyhedra. Thus, one can similarly see that (ii) also follows from Theorems 3.3 and 3.2. The proof is completed. ∎

In the case when the ordering cone C is polyhedral, we have the following sharper results (with $d(V_w, F(x) + C)$ replacing $d(V_w, F(x))$).

THEOREM 4.2. *Suppose that C is polyhedral and has a nonempty interior. Then, the following statements hold:*

(i) *For any* $r \in (0, +\infty)$ *there exists* $\eta \in (0, +\infty)$ *such that*

$$\eta d(x, S_w) \leq d(V_w, F(x) + C) + d(x, \Gamma) \quad \forall x \in S + rB_X;$$

(ii) *there exists* $\eta \in (0, +\infty)$ *such that*

$$\eta d(x, S_w) \leq d(V_w, F(x) + C) + d(x, \Gamma) \quad \forall x \in X$$

if and only if

$$\lim_{d(x,S_w)\to\infty} d(V_w, F(x) + C) + d(x, P) = +\infty.$$

Proof. Let

$$\tilde{F}(x) := F(x) + C \ \ \forall x \in X.$$

Then $\text{Gr}(\tilde{F}) = \text{Gr}(F) + (\{0\} \times C)$. Since F is a piecewise linear multifunction, it is easy from Lemma 2.1 to verify that $\text{Gr}(\tilde{F})$ is the union of finitely many polyhedra in $X \times Y$. We claim that

$$S_w = \Gamma \cap \tilde{F}^{-1}(V_w). \tag{4.1}$$

Granting this, (i) and (ii) are immediate from Lemma 2.7 and Theorems 3.2 and 3.3. It remains to show that (4.1) holds. By the definition of $\tilde{F}$, one has $F(x) \subset \tilde{F}(x)$ for all $x \in X$ and so $S_w = \Gamma \cap F^{-1}(V_w) \subset \Gamma \cap \tilde{F}^{-1}(V_w)$. We need only show the converse inclusion. Let $x \in \Gamma \cap \tilde{F}^{-1}(V_w)$. Then, there exists $y \in V_w$ such that $y \in \tilde{F}(x)$ and so there exists $y' \in F(x)$ such that $y \in y' + C$. Hence $F(\Gamma) \cap (y - \text{int}(C)) = \emptyset$ and $y' - \text{int}(C) \subset y - \text{int}(C)$. It follows that $F(\Gamma) \cap (y' - \text{int}(C)) = \emptyset$, this is, $y' \in V_w = \text{WE}(F(\Gamma), C)$. Hence, $x \in \Gamma \cap F^{-1}(y') \subset \Gamma \cap F^{-1}(V_w) = S_w$. The proof is completed. □

Remark. In general, Theorem 4.2 is not true if the weak-Pareto solution set S_w and the weak-Pareto optimal value set V_w are replaced by the Pareto solution set S and the Pareto optimal value set V, respectively. Indeed, let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Gamma = X$, $C = \mathbb{R}^2_+$, and

$$F(x) = \begin{cases} (x, 0), & \text{if } x \leq 0, \\ (x, -x), & \text{if } x \geq 0. \end{cases}$$

Then $F(\Gamma) = (\mathbb{R}_- \times \{0\}) \cup \mathbb{R}_+(1, -1)$. Hence, $V = \text{E}(F(\Gamma), C) = \{(x, -x) : \ x > 0\}$ and so $S = F^{-1}(V) = (0, +\infty)$. It follows that $d(V, F(x) + C) = 0$ for all $x \in X$ and $d(x, S) = |x|$ for all $x \in (-\infty, 0)$. This shows that Theorem 4.2 is not true if S_w and V_w are replaced by S and V.

Under the C-convexity assumption on the objective F, we can establish an interesting result that (1.3) always has the global weak sharp minimum property with respect to weak-Pareto optimal values for (1.3). Recall that F is C-convex if its C-epigraph $\text{epi}_C(F)$ is a convex subset of $X \times Y$, where

$$\text{epi}_C(F) := \{(x, y) : \ x \in X \text{ and } y \in F(x) + C\}.$$

It is clear and known that F is C-convex if and only if

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + C \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0, 1].$$

In fact, under the C-convexity assumption, we have the following sharper result.

THEOREM 4.3. *Let the ordering cone C have nonempty interior and suppose that F is C-convex. Then there exists $\eta \in (0, +\infty)$ such that*

$$\eta d(x, S_w) \leq d(V_w, F(x) + C) + d(x, \Gamma) \quad \forall x \in X. \tag{4.2}$$

Consequently, (1.3) *has the global weak sharp minimum property with respect to weak-Pareto optimal values for* (1.3).

Proof. Since F is C-convex, $F(\Gamma) + C$ is convex. Hence Lemma 2.6(i) is applicable and there exists $y_i^* \in C^+$ with $\|y_i^*\| = 1$ $(1 \leq i \leq q)$ such that (2.4) holds: $V_w = \bigcup_{i=1}^q L(y_i^*)$, where we may assume without loss of generality that each $L(y_i^*)$ is nonempty. We consider an arbitrary i $(1 \leq i \leq q)$ and keep it fixed. Let $F_i : X \rightrightarrows \mathbb{R}$ be defined by

$$F_i(x) = \{\langle y_i^*, y\rangle : \ y \in F(x)\} + \mathbb{R}_+ \ \ \forall x \in X, \tag{4.3}$$

that is,

$$\mathrm{Gr}(F_i) = \{(x, \langle y_i^*, y\rangle) : (x, y) \in \mathrm{Gr}(F)\} + \{0\} \times \mathbb{R}_+. \tag{4.4}$$

Then $\mathrm{Gr}(F_i)$ is convex, thanks to the C-convexity assumption on F and the fact that $y_i^* \in C^+$. We claim that it is a polyhedron. To do this, by virtue of Lemma 2.1 and [31, Lemma 2.50], it suffices to show that $\mathrm{Gr}(F_i)$ is the union of finitely many polyhedra. For this, we note that, since F is a piecewise linear multifunction, there exist polyhedra $\mathcal{P}_1, \dots, \mathcal{P}_m$ in $X \times Y$ such that $\mathrm{Gr}(F) = \bigcup_{l=1}^m \mathcal{P}_l$. For each l, we apply Lemma 2.1: there exist closed subspaces X_l and X_l' of X, closed subspaces Y_l and Y_l' of Y, and a polyhedron $\mathcal{P}_l'$ in $X_l' \times Y_l'$ such that X_l' and Y_l' are finite dimensional,

$$X \times Y = (X_l \times Y_l) \oplus (X_l' \times Y_l'), \quad \text{and} \quad \mathcal{P}_l = X_l \times Y_l + \mathcal{P}_l'.$$

Consider the linear operator $T_i : X \times Y \to X \times \mathbb{R}$ defined by

$$T_i(x, y) = (x, \langle y_i^*, y\rangle) \quad \forall (x, y) \in X \times Y.$$

Then, by [30, Theorem 19.3], $T_i(\mathcal{P}_l')$ is a polyhedron in the finite dimensional space $X_l' \times \mathbb{R}$, and so $T_i(\mathcal{P}_l') + \{0\} \times (y_i^*(Y_l) + \mathbb{R}_+)$ is a polyhedron in $X_l' \times \mathbb{R}$. Noting that

$$\begin{aligned} T_i(\mathcal{P}_l) + \{0\} \times \mathbb{R}_+ &= X_l \times y_i^*(Y_l) + T_i(\mathcal{P}_l') + \{0\} \times \mathbb{R}_+ \\ &= X_l \times \{0\} + [T_i(\mathcal{P}_l') + \{0\} \times (y_i^*(Y_l) + \mathbb{R}_+)], \end{aligned}$$

this and Lemma 2.1 imply that $T_i(\mathcal{P}_l) + \{0\} \times \mathbb{R}_+$ is a polyhedron in $X \times \mathbb{R}$ (because $X \times \mathbb{R} = (X_l \times \{0\}) \oplus (X_l' \times \mathbb{R})$ and $X_l' \times \mathbb{R}$ is finite dimensional). Noting (by (4.4)) that

$$\mathrm{Gr}(F_i) = T_i(\mathrm{Gr}(F)) + \{0\} \times \mathbb{R}_+ = \bigcup_{l=1}^m (T_i(\mathcal{P}_l) + \{0\} \times \mathbb{R}_+),$$

it follows that $\mathrm{Gr}(F_i)$ is the union of finitely many polyhedra in $X \times Y$ (and so itself is a polyhedron by the convexity as noted). Noting that $X \times \{\lambda_{y_i^*}\}$ is a polyhedron in $X \times \mathbb{R}$ and that $F_i^{-1}(\lambda_{y_i^*}) = \big(\mathrm{Gr}(F_i) \cap (X \times \{\lambda_{y_i^*}\})\big)_X$, where $\lambda_{y_i^*}$ is as in (2.3), this and Lemma 2.2 imply that $F_i^{-1}(\lambda_{y_i^*})$ is a polyhedron in X. Thus, Lemmas 2.4 and 2.3 are applicable: there exist $\eta_i', \eta_i'' \in (0, +\infty)$ such that

$$\eta_i' d(x, \Gamma \cap F_i^{-1}(\lambda_{y_i^*})) \leq d(x, F_i^{-1}(\lambda_{y_i^*})) + d(x, \Gamma) \quad \forall x \in X$$

and

$$\eta_i'' d(x, F_i^{-1}(\lambda_{y_i^*})) \le d(\lambda_{y_i^*}, F_i(x)) \quad \forall x \in X.$$

It follows that

$$\eta_i d(x, \Gamma \cap F_i^{-1}(\lambda_{y_i^*})) \le d(\lambda_{y_i^*}, F_i(x)) + d(x, \Gamma) \quad \forall x \in X, \tag{4.5}$$

where $\eta_i = \frac{\eta_i' \eta_i''}{\max\{1, \eta_i''\}}$. We claim that

$$\Gamma \cap F_i^{-1}(\lambda_{y_i^*}) = \Gamma \cap F^{-1}(L(y_i^*)) \quad \forall i \in \overline{1, q} \tag{4.6}$$

and that

$$d(\lambda_{y_i^*}, F_i(x)) \le d(L(y_i^*), F(x) + C) \quad \forall (x, i) \in X \times \overline{1, q}. \tag{4.7}$$

Granting these, (4.5) implies that

$$\eta_i d(x, \Gamma \cap F^{-1}(L(y_i^*))) \le d(L(y_i^*), F(x) + C) + d(x, \Gamma) \quad \forall (x, i) \in X \times \overline{1, q}$$

and hence that, by taking infimum over all i in $\overline{1, q}$,

$$\bar{\eta} d\left(x, \Gamma \cap \bigcup_{i=1}^{q} F^{-1}(L(y_i^*))\right) \le d\left(\bigcup_{i=1}^{q} L(y_i^*), F(x) + C\right) + d(x, \Gamma) \quad \forall x \in X,$$

where $\bar{\eta} = \min\{\eta_i : 1 \le i \le q\}$. This means that (4.2) holds with $\eta = \bar{\eta}$ (since $S_w = \Gamma \cap F^{-1}(V_w)$ by definition and $V_w = \bigcup_{i=1}^{q} L(y_i^*)$ by (2.4)).

It remains to verify our claims (4.6) and (4.7). Let $i \in \overline{1, q}$. Let $x \in \Gamma \cap F_i^{-1}(\lambda_{y_i^*})$, that is, $x \in \Gamma$ and $\lambda_{y_i^*} \in F_i(x)$. Then, by the definition of F_i, there exists $y \in F(x)$ such that $\lambda_{y_i^*} \ge \langle y_i^*, y\rangle$ and so $\lambda_{y_i^*} = \langle y_i^*, y\rangle$ (by the definition of $\lambda_{y_i^*}$ and the fact that $x \in \Gamma$). Hence $y \in L(y_i^*)$ and so $x \in \Gamma \cap F^{-1}(L(y_i^*))$. This implies that (4.6) holds as it is easy to check that $F_i^{-1}(\lambda_{y_i^*}) \supset F^{-1}(L(y_i^*))$. For (4.7), let $y \in F(x)$, $c \in C$, and $z \in L(y_i^*)$. Then $\langle y_i^*, z\rangle = \lambda_{y_i^*}$ and $\langle y_i^*, y + c\rangle \in F_i(x)$ (since $y_i^* \in C^+$). Hence

$$d(\lambda_{y_i^*}, F_i(x)) \le |\langle y_i^*, z\rangle - \langle y_i^*, y + c\rangle| \le \|z - y - c\|$$

and so (4.7) holds. The proof is complete. □

Similarly to the above proof but with Lemma 2.6(i) replaced by Lemma 2.6(ii), we can prove the following result.

THEOREM 4.4. *Let the ordering cone C have a weakly compact base and suppose that F is C-convex. Then there exists $\eta \in (0, +\infty)$ such that*

$$\eta d(x, S) \le d(V, F(x) + C) + d(x, \Gamma) \;\; \forall x \in X. \tag{4.8}$$

Consequently, (1.3) *has the global weak sharp minimum property with respect to Pareto optimal values for* (1.3).

Note that every closed convex pointed cone in a finite dimensional space has a compact base. Thus the following corollary is immediate from Theorem 4.4.

COROLLARY 4.1. *Let Y be finite dimensional and the ordering cone C be pointed. Suppose that F is C-convex. Then there exists $\eta \in (0, +\infty)$ such that* (4.8) *holds.*

It is easy to provide examples showing that Theorems 4.3 and 4.4 are not true (even in the finite dimensional case) if the C-convexity of F is dropped.

Acknowledgment. The authors wish to thank the referees for their helpful comments which improve the presentation of the paper.

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