METRIC SUBREGULARITY AND CONSTRAINT QUALIFICATIONS FOR CONVEX GENERALIZED EQUATIONS IN BANACH SPACES

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Abstract. Several notions of constraint qualifications are generalized from the setting of convex inequality systems to that of convex generalized equations. This is done and investigated in terms of the coderivatives and the normal cones, and thereby we provide some characterizations for convex generalized equations to have the metric subregularity. As applications, we establish formulas of the modulus of calmness and provide several characterizations of the calmness. Extending the classical concept of extreme boundary, we introduce a notion of recession cores of closed convex sets. Using this concept, we establish global metric subregularity (i.e. error bound) results for generalized equations.

Key words. *metric subregularity, calmness, constraint qualification, normal cone, coderivative, recession core, generalized extreme point*

AMS subject classifications. *90C31, 90C25, 49J52, 46B20*

1. Introduction. Let X and Y be Banach spaces and $F : X \to 2^Y$ a closed multifunction. Following Dontchev and Rockafellar [7], the multifunction F is said to be metrically subregular at a for $b \in F(a)$ if there exists $\tau \in [0, +\infty)$ such that

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) \ \text{ for all } x \text{ close } to \ a. \tag{1.1}$$

The metric subregularity has been already studied by many authors under various names (cf. [2,15,20,24,30]). Let A be a closed subset of X and b be a given point in Y. Consider the generalized equation with constraint

(GEC) $\qquad\qquad\qquad b \in F(x) \ \text{ subject to } x \in A,$

which includes most of systems in optimization. Let S denote the solution set of (GEC), that is, $S = \{x \in A : \ b \in F(x)\}$. We say that (GEC) is metrically subregular at $a \in S$ if there exists $\tau \in (0, \infty)$ such that

$$d(x, S) \leq \tau(d(b, F(x)) + d(x, A)) \ \text{ for all } x \text{ close to } a. \tag{1.2}$$

When $F(x) = [f(x), +\infty)$, $b = 0$ and $A = X$, (GEC) reduces to the inequality system $f(x) \leq 0$ and (1.2) means that this inequality has a local error bound at a. Another

special case is when $F(x) = [f(x), +\infty)$ and $b = \inf\{f(x) : x \in A\}$. In this case, (GEC) reduces to the following optimization problem

(OP) $$\min f(x) \text{ subject to } x \in A$$

and (1.2) means that a is weak sharp minimum of (OP). Error bounds and weak sharp minima have important applications in mathematical programming and have been extensively studied (cf. [3,17,18,20,30,31]). In this paper, we mainly study the metric subregularity of (GEC) in the case when F and A are convex.

The notion of the basic constraint qualification (BCQ) of systems of continuous convex inequalities plays an important role in convex optimization and has been studied by many researchers (see, e.g., [11, pp.307-309] and [18,19]). Dropping the continuity assumption and adopting the singular subdifferential, the authors [32] introduced and discussed the generalized BCQ and the strong BCQ. Very recently, Hu [12] further studied the generalized BCQ and the strong BCQ. In Section 3, in terms of the coderivative, we extend the concept of the generalized BCQ and the strong BCQ to cover the case of generalized equation with constraint (GEC). Using the BCQ and the strong BCQ, we provide several characterizations of the metric subregularity of (GEC).

A stronger condition is the metric regularity of a multifunction that has been well studied in variational analysis (see [13,15,21,28] and references therein). Explicitly, F is metrically regular at a for $b \in F(a)$ if there exists $\tau \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \text{ for all } (x, y) \text{ close } to (a, b). \tag{1.3}$$

It is well-known, as the Robinson-Ursescu theorem, that (1.3) holds if F is a closed convex multifunction and $b \in \text{int}(F(X))$. Under the assumption that both X and Y are finite dimensional, Mordukhovich [21] proved that F is metrically regular at a for $b \in F(a)$ if and only if $D^*F(a, b)^{-1}(0) = \{0\}$; moreover

$$\inf\{\tau : (1.3) \text{ holds}\} = \|D^*F(a, b)^{-1}\|^+ = \limsup_{(x,y) \xrightarrow{\text{Gr}(F)} (a,b)} \|D^*F(x, y)^{-1}\|^+, \tag{1.4}$$

where $D^*F(a, b)$ is the coderivative of F at (a, b) and

$$\|D^*F(a, b)^{-1}\|^+ = \sup_{x^* \in B_{X^*}} \sup_{y^* \in D^*F(a,b)^{-1}(x^*)} \|y^*\|.$$

To the best of our knowledge, no one has considered duality formulas similar to (1.4) for the modulus of the metric subregularity. In Section 3, we provide such formulas under the convexity assumption but no finite dimension assumption.

Similar to relationship between Aubin's pseudo-Lipschitz property and the metric regularity, the calmness is related very closely to the metric subregularity. In Section 4, as applications of results obtained in Section 3, we consider the calmness of convex multifunctions. We establish formulas of the modulus of the calmness and

present several characterizations of the calmness in terms of the normal cone and the coderivative. In this section, we also provide characterizations of the strong calmness. Reducing to special kinds of convex multifunctions such as that recently considered by Henrion and Jourani in [8], our approach sheds light on some existing results on the calmness; in fact Corollary 4.1 provides a version that is sharper than the main result in [8].

The notion of the extreme point set of a convex set is very useful in convex analysis. In Section 5, as an extension of an extreme point set, we introduce and discuss the notion of a recession core. In terms of recession cores, we study the global metric subregularity. In particular, we show that (GEC) is globally metrically subregular if and only if (GEC) has the τ-strong BCQ at each point of some recession core of the solution set S for some $\tau \in (0, +\infty)$. When the solution set S is a polyhedron, we obtain a sharp result that (GEC) is globally metrically subregular if and only if (GEC) has the BCQ at each point of some recession core of S. This implies in particular that if the graph of F is a polyhedron then (GEC) is always globally metrically subregular; thus we improve the classical Hoffman result on error bound for linear inequality systems.

2. Preliminaries. Throughout this paper, we assume that X and Y are Banach spaces. Let A be a closed convex subset of X. For $a \in A$, we use $T(A, a)$ to denote the tangent cone of A at a in the sense of convex analysis. Thus $v \in T(A, a)$ if and only if there exist a sequence $\{a_n\}$ in A and a sequence $\{t_n\}$ of positive numbers convergent to 0 such that $\frac{a_n - a}{t_n}$ converges to v. We denote by $N(A, a)$ the normal cone of A at a, that is,

$$N(A, a) := \{x^* \in X^* : \langle x^*, x - a \rangle \leq 0 \text{ for all } x \in A\}.$$

Let $F : X \to 2^Y$ be a multifunction and denote by $\mathrm{Gr}(F)$ the graph of F, that is,

$$\mathrm{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

As usual, F is said to be closed (resp. convex) if $\mathrm{Gr}(F)$ is a closed (resp. convex) subset of $X \times Y$. It is known that F is convex if and only if

$$tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0, 1].$$

Let F be a closed convex multifunction and $(x, y) \in \mathrm{Gr}(F)$. The tangent derivative $DF(x, y)$ of F at (x, y) is defined by

$$DF(x, y)(u) = \{v \in Y : (u, v) \in T(\mathrm{Gr}(F), (x, y))\} \quad \forall u \in X \qquad (2.1)$$

(cf. [1]). Let $D^*F(x, y)$ denote the coderivative of F at (x, y), which is defined by

$$D^*F(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N(\mathrm{Gr}(F), (x, y))\} \quad \forall y^* \in Y^* \qquad (2.2)$$

(cf. [21,22,28]). Let $G : X \to 2^Y$ be a sublinear multifunction (i.e., $\mathrm{Gr}(G)$ is a convex cone in $X \times Y$). Let B_X denote the closed unit ball of X. As in Dontchev, Lewis and

Rockafellar [6], the outer norm and the inner norm of G are respectively defined as follows

$$\|G\|^+ = \sup_{x \in B_X} \sup_{y \in Gx} \|y\| \quad \text{and} \quad \|G\|^- := \sup_{x \in B_X} \inf_{y \in Gx} \|y\|,$$

with the usual convention that the infimum and the supremum over an empty set are $+\infty$ and $-\infty$, respectively. For a convex cone C in X, let $\|G|_C\|^+$ and $\|G|_C\|^-$ be respectively defined by

$$\|G|_C\|^+ := \sup_{x \in B_X \cap C} \sup_{y \in Gx} \|y\| \quad \text{and} \quad \|G|_C\|^- := \sup_{x \in B_X \cap C} \inf_{y \in Gx} \|y\|. \qquad (2.3)$$

We denote by $\mathrm{bd}(A)$ the topological boundary of a subset A of X. The following lemma is known (cf. [23, Proposition 1.3] or [24, Lemma 2.1]) and useful for us.

LEMMA 2.1. *Let X be a Banach space and A a closed convex nonempty subset of X. Then, for any $\beta \in (0, 1)$ and any $x \in X \setminus A$ there exist $z \in \mathrm{bd}(A)$ and $x^* \in N(A, z)$ with $\|x^*\| = 1$ such that*

$$\beta\|x - z\| < d(x, A) \quad \text{and} \quad \beta\|x - z\| < \langle x^*, x - z \rangle.$$

3. BCQ, strong BCQ and metric subregularity. Throughout this section, we assume that $F : X \to 2^Y$ is a closed convex multifunction, A is a closed convex subset of X and that b is a given point in X. Recall that $S = \{x \in A : b \in F(x)\}$ is the solution set of the corresponding generalized equation with constraint (GEC).

Recently, in dealing with the inequality defined by a proper lower semicontinuous convex function, the authors [32] introduced and discussed the generalized BCQ and the strong BCQ.

In terms of the coderivative replacing the subdifferential and the singular subdifferential, we can extend the concept of the generalized BCQ and the strong BCQ to the case of generalized equation with constraint (GEC). Explicitly, we say that (GEC) has the BCQ at $a \in S$ if

$$N(S, a) = D^*F(a, b)(Y^*) + N(A, a) \qquad (3.1)$$

and (GEC) has the strong BCQ at $a \in S$ if there exists $\tau \in (0, +\infty)$ such that

$$N(S, a) \cap B_{X^*} \subset \tau(D^*F(a, b)(B_{Y^*}) + B_{X^*} \cap N(A, a)). \qquad (3.2)$$

The following Theorem establishes relationship between the metric subregularity and the strong BCQ.

THEOREM 3.1. *Let $a \in S$. Then, generalized equation (GEC) is metrically subregular at a if and only if there exist $\tau, \delta \in (0, +\infty)$ such that (GEC) has the strong BCQ at all points in $\mathrm{bd}(S) \cap B(a, \delta)$ with the same constant, where $B(a, \delta)$ denotes the open ball with center a and radius δ.*

Proof. Suppose that (GEC) is metrically subregular at a. Then there exist $\tau, \delta \in (0, +\infty)$ such that (1.2) holds. For any $(x, y) \in X \times Y$, let $\|(x,y)\|_\tau := \frac{\tau+1}{\tau}\|x\| + \|y\|$. Then $\|\cdot\|_\tau$ is a norm on $X \times Y$ inducing the product topology, and the unit ball of the dual space of $(X \times Y, \|\cdot\|_\tau)$ is $(\frac{\tau}{\tau+1}B_{X^*}) \times B_{Y^*}$. We claim that

$$d(x, S) \leq \tau(d_{\|\cdot\|_\tau}((x,y), \mathrm{Gr}(F)) + \|y - b\| + d(x, A)) \quad \forall (x, y) \in B(a, \frac{\delta}{2}) \times Y, \quad (3.3)$$

where the distance $d_{\|\cdot\|_\tau}$ is with respect to the norm $\|\cdot\|_\tau$. Suppose to the contrary that (3.3) does not hold. Then there exists $(x_0, y_0) \in B(a, \frac{\delta}{2}) \times Y$ such that

$$d(x_0, S) > \tau[d_{\|\cdot\|_\tau}((x_0, y_0), \mathrm{Gr}(F)) + \|y_0 - b\| + d(x_0, A)].$$

It follows that there exists $u \in X$ such that

$$d(x_0, S) > \tau\left(\frac{\tau+1}{\tau}\|u - x_0\| + d(y_0, Fu) + \|y_0 - b\| + d(x_0, A)\right),$$

and hence

$$d(x_0, S) > \|u - x_0\| + \tau(d(b, Fu) + d(u, A)).$$

Noting that

$$\|u - a\| \leq \|u - x_0\| + \|x_0 - a\| < d(x_0, S) + \|x_0 - a\| \leq 2\|x_0 - a\| < \delta,$$

it follows from (1.2) and the triangle inequality that

$$d(x_0, S) > \|u - x_0\| + d(u, S) \geq d(x_0, S),$$

which is a contradiction. Hence (3.3) holds. We will establish the necessary part by showing that

$$N(S, z) \cap B_{X^*} \subset \tau(D^*F(z, b)(B_{Y^*}) + B_{X^*} \cap N(A, z)) \quad \forall z \in B(a, \frac{\delta}{2}) \cap S. \quad (3.4)$$

To do this, let $z \in S \cap B(a, \frac{\delta}{2})$ and $x^* \in B_{X^*} \cap N(S, z)$. Since A is convex, $B_{X^*} \cap N(S, z) = \partial d(\cdot, S)(z)$ (cf. [4, Theorem 1]). Thus,

$$\langle x^*, x - z \rangle \leq d(x, S) - d(z, S) = d(x, S) \text{ for all } x \in X.$$

It follows from (3.3) that

$$\langle x^*, x - z \rangle \leq \tau(d_{\|\cdot\|_\tau}(x, y), \mathrm{Gr}(F)) + \|y - b\| + d(x, A)) \quad \forall (x, y) \in B(z, \frac{\delta}{2} - \|z - a\|) \times Y.$$

Together with the convexity of F and A, this implies that $(\frac{x^*}{\tau}, 0) \in \partial\phi(z, b)$, where ϕ is the convex function defined by

$$\phi(x, y) := d_{\|\cdot\|_\tau}((x, y), \mathrm{Gr}(F)) + \|y - b\| + d(x, A) \quad \forall (x, y) \in X \times Y.$$

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Noting that
$$\partial d_{\|\cdot\|_\tau}(\cdot, \mathrm{Gr}(F))(z, b) \subset N(\mathrm{Gr}(F), (z, b)),$$

it follows from [5, Proposition 2.3.2] that

$$(\frac{x^*}{\tau}, 0) \in N(\mathrm{Gr}(F), (z, b)) + \{0\} \times B_{Y^*} + (B_{X^*} \cap N(A, z)) \times \{0\}.$$

This implies that $x^* \in \tau(D^*F(z, b)(B_{Y^*}) + B_{X^*} \cap N(A, z))$, and hence that (3.4) holds as required to show.

Conversely, suppose that there exist $\tau', \delta' \in (0, +\infty)$ such that (GEC) has the strong BCQ at each point of $\mathrm{bd}(S) \cap B(a, \delta')$ with the constant τ'. Let $x \in B(a, \frac{\delta'}{2}) \backslash S$. Then, $d(x, S) \leq \|x - a\| < \frac{\delta'}{2}$. Let $\beta \in (\frac{2d(x,S)}{\delta'}, 1)$. Then, by Lemma 2.1 there exists $u \in \mathrm{bd}(S)$ and $x^* \in N(S, u)$ with $\|x^*\| = 1$ such that $\beta\|x - u\| \leq d(x, S)$ and

$$\beta\|x - u\| \leq \langle x^*, x - u \rangle. \tag{3.5}$$

Thus, $\|x - u\| < \frac{\delta'}{2}$. Hence $\|u - a\| \leq \|u - x\| + \|x - a\| < \delta'$, and so (GEC) has the strong BCQ at u with the constant τ. Therefore, there exists $y^* \in B_{Y^*}$, $x_1^* \in D^*F(u, b)(y^*)$ and $x_2^* \in B_{X^*} \cap N(A, u) = \partial d(\cdot, A)(u)$ (by [4,Theorem 1]) such that $x^* = \tau'(x_1^* + x_2^*)$. By the convexity of F and A, one has

$$\langle x_1^*, x - u \rangle \leq \langle y^*, y - b \rangle \quad \forall y \in F(x) \text{ and } \langle x_2^*, x - u \rangle \leq d(x, A) - d(u, A) = d(x, A).$$

Hence,

$$\langle x^*, x - u \rangle \leq \tau'(\langle y^*, y - b \rangle + d(x, A)) \leq \tau'(\|y - b\| + d(x, A)) \quad \forall y \in F(x).$$

This and (3.5) imply that $\beta\|x - u\| \leq \tau'(d(b, F(x)) + d(x, A))$. It follows from $u \in S$ that $\beta d(x, S) \leq \tau'(d(b, F(x)) + d(x, A))$. Since β can be arbitrarily close to 1, $d(x, S) \leq \tau'(d(b, F(x)) + d(x, A))$. This shows that (GEC) is metrically subregular at a. This completes the proof.

Theorem 3.1 recaptures some earlier results dealing with only numerical valued functions. Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous convex function. When $F(x) = [f(x), +\infty)$, $b = 0$ and $A = X$, Theorem 3.1 was obtained in [32]. A slightly earlier result is due to Burke and Deng who showed in [3, Theorem 5.2] that if X is a Hilbert space, $F(x) = [f(x), +\infty)$, $b = \inf_{x \in X} f(x)$ and $A = X$, then (GEC) is metrically subregular at a if and only if there exists $\tau \in [0, +\infty)$ such that

$$N(S, x) \cap B_{X^*} \subset \tau \mathrm{cl}^*(\partial f(a)),$$

where cl^* denotes the weak* closure.

Remark 3.1. Let $\tau(F, a, b; A) := \inf\{\tau > 0 : (1.2) \text{ holds}\}$. For $u \in S$, let

$$\gamma(F, u, b; A) := \inf\{\tau > 0 : (\mathrm{GEC}) \text{ has the strong BCQ at } u \text{ with the constant } \tau\}.$$

By the proof of Theorem 3.1, one can see that

$$\tau(F, a, b; A) = \limsup_{u \overset{\mathrm{bd}(S)}{\longrightarrow} a} \gamma(F, u, b; A) \geq \gamma(F, a, b; A). \qquad (3.6)$$

In general, (GEC) is not necessarily metrically subregular at a if (GEC) has the strong BCQ only at a (see [32, Example 2]). But, when S is assumed to be "locally conical" at a, Theorem 3.1 and (3.6) can be sharpened. To do this, we need the following lemma.

LEMMA 3.1. *Let* $s_1, s_2 \in S$ *be such that* $\langle u^*, s_1 \rangle = \langle u^*, s_2 \rangle$. *Then,*

$$u^* \in D^* F(s_1, b)(B_{Y^*}) + B_{X^*} \cap N(A, s_1) \Leftrightarrow u^* \in D^* F(s_2, b)(B_{Y^*}) + B_{X^*} \cap N(A, s_2).$$

Proof. Obviously we need only to prove one direction of the implications, say " $\Rightarrow$ ". Let

$$\psi(x, y) := \|y - b\| + d(x, A) + \delta_{\mathrm{Gr}(F)}(x, y) \text{ for all } (x, y) \in X \times Y,$$

where $\delta_{\mathrm{Gr}(F)}$ denotes the indicator function of $\mathrm{Gr}(F)$. It follows from [4, Theorem 1] and [5, Proposition 2.3.2] that

$$\partial \psi(s, b) = \{0\} \times B_{Y^*} + (B_{X^*} \cap N(A, s)) \times \{0\} + N(\mathrm{Gr}(F), (s, b)) \quad \forall s \in S. \quad (3.7)$$

Suppose that $u^* \in D^* F(s_1, b)(B_{Y^*}) + B_{X^*} \cap N(A, s_1)$. Then, by (3.7), one has $(u^*, 0) \in \partial \psi(s_1, b)$. Hence,

$$\langle u^*, x - s_1 \rangle \leq \psi(x, y) - \psi(s_1, b) \text{ for all } (x, y) \in X \times Y.$$

Since $\langle u^*, s_1 \rangle = \langle u^*, s_2 \rangle$ and $\psi(s_1, b) = \psi(s_2, b) = 0$,

$$\langle u^*, x - s_2 \rangle \leq \psi(x, y) - \psi(s_2, b) \text{ for all } (x, y) \in X \times Y.$$

Therefore, $(u^*, 0) \in \partial \psi(s_2, b)$. It follows from (3.7) that $u^* \in D^* F(s_2, b)(B_{Y^*}) + B_{X^*} \cap N(A, s_2)$. This shows that the implication " $\Rightarrow$ " holds. The proof is completed.

THEOREM 3.2. *Let* $a \in S$. *Suppose that there exist a cone* C *and a neighborhood* V *of* a *such that* $S \cap V = (a + C) \cap V$. *Then*

$$\tau(F, a, b; A) = \gamma(F, a, b; A).$$

Consequently, (GEC) is metrically subregular at a *if and only if (GEC) has the strong BCQ at* a.

Proof. In view of (3.6), we need only to show that

$$\limsup_{u \overset{\mathrm{bd}(S)}{\longrightarrow} a} \gamma(F, u, b; A) \leq \gamma(F, a, b; A). \qquad (3.8)$$

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Let $\delta > 0$ be such that $B(a, \delta) \subset V$. To prove (3.8), it suffices to show that for any $u \in S \cap B(a, \delta)$,

$$\gamma(F, u, b; A) \leq \gamma(F, a, b; A). \tag{3.9}$$

We first show that

$$N(S, u) \subset N(S, a) \text{ for all } u \in S \cap B(a, \delta). \tag{3.10}$$

To do this, let $u \in S \cap B(a, \delta)$ and $x^* \in N(S, u)$. Noting that V is a neighborhood of u, we have

$$N(S, u) = N(S \cap V, u) = N((a + C) \cap V, u) = N(a + C, u).$$

Choosing $c_u \in C$ such that $u = a + c_u$, it follows that

$$\langle x^*, a + c_u \rangle = \sup\{\langle x^*, a + c \rangle : c \in C\}.$$

Since C is a cone, it follows that $\langle x^*, c_u \rangle = 0$ and hence

$$\langle x^*, u \rangle = \langle x^*, a \rangle = \sup\{\langle x^*, a + c \rangle : c \in C\}. \tag{3.11}$$

This implies that $x^* \in N(a + C, a) = N(S, a)$. Therefore, (3.10) holds. Since (3.9) trivially holds if $\gamma(F, a, b; A) = +\infty$, we assume henceforth that $\gamma(F, a, b; A) < +\infty$. Let $r \in (\gamma(F, a, b; A), +\infty)$. Then,

$$B_{X^*} \cap N(S, a) \subset r(D^*F(a, b)(B_{Y^*}) + B_{X^*} \cap N(A, a)).$$

Let $u \in S \cap B(a, \delta)$ and $x^* \in B_{X^*} \cap N(S, u)$. By (3.10), one has

$$x^* \in r(D^*F(a, b)(B_{Y^*}) + B_{X^*} \cap N(A, a)).$$

It follows from (3.11) and Lemma 3.1 that $x^* \in r(D^*F(u, b)(B_{Y^*}) + B_{X^*} \cap N(A, u))$. Therefore,

$$B_{X^*} \cap N(S, u) \subset r(D^*F(u, b)(B_{Y^*}) + B_{X^*} \cap N(A, u)).$$

This implies that $\gamma(F, u, b; A) \leq r$. Letting $r \to \gamma(F, a, b; A)$, one sees that (3.9) holds. The proof is completed.

Remark 3.2. If the solution set S is a polyhedron then for each $a \in S$ there exist a cone C and a neighborhood V of a such that $S \cap V = (a + C) \cap V$; in fact, in this case we can choose C to be the tangent cone of S at a.

THEOREM 3.3. *Let* $a \in S$,

$$\tau_1 := \inf\{\tau > 0 : d(x, a + T(S, a)) \leq \tau(d(b, F(x)) + d(x, A)) \text{ for all } x \text{ close to } a\}$$

and

$$\tau_2 := \inf\{\tau > 0 : \ d(h, T(S, a)) \le \tau(d(0, DF(a, b)(h)) + d(h, T(A, a))) \quad \forall h \in X\}.$$

Then

$$\tau_1 = \tau_2 = \gamma(F, a, b; A).$$

Moreover,

$$\tau_2 < +\infty \Longrightarrow T(S, a) = T(A, a) \cap DF(a, b)^{-1}(0). \tag{3.12}$$

Consequently, (GEC) has the strong BCQ at a if and only if the sublinear generalized equation (with constraint)

$$0 \in DF(a, b)(x) \ \text{subject to} \ x \in T(A, a)$$

is metrically subregular at 0.

Proof. We first show that $\tau_1 = \tau_2$. Let $h \in X$, $y \in DF(a, b)(h)$, $u \in T(A, a)$ and $\varepsilon > 0$. Then, there exists $t > 0$ small enough such that

$$(h, y) \in \frac{\mathrm{Gr}(F) - (a, b)}{t} + \varepsilon B_X \times \varepsilon B_Y \ \text{and} \ u \in \frac{A - a}{t} + \varepsilon B_X.$$

Therefore, there exists $z \in B_X$ such that

$$b + ty \in F(a + th + t\varepsilon z) + t\varepsilon B_Y \ \text{and} \ a + tu \in A + t\varepsilon B_X.$$

This implies that

$$d(b, F(a + th + t\varepsilon z)) \le t\|y\| + t\varepsilon \ \text{and} \ d(a + th + t\varepsilon z, A) \le t\|h - u\| + 2t\varepsilon.$$

Considering an arbitrary $\tau > \tau_1$ and noting that $t > 0$ is small enough, it follows that

$$\begin{aligned}
\tau t(\|y\| + \|h - u\| + 3\varepsilon) &\ge d(a + th + t\varepsilon z, a + T(S, a)) \\
&\ge d(th, T(S, a)) - t\varepsilon \\
&= td(h, T(S, a)) - t\varepsilon,
\end{aligned}$$

where the last equality holds because $T(S, a)$ is a cone. Therefore,

$$d(h, T(S, a)) \le \tau(d(0, DF(a, b)(h)) + d(h, T(A, a))) + (3\tau + 1)\varepsilon.$$

Letting $\varepsilon \to 0$ and $\tau \to \tau_1$, one has

$$d(h, T(S, a)) \le \tau_1(d(0, DF(a, b)(h)) + d(h, T(A, a))).$$

Hence $\tau_2 \le \tau_1$.

Conversely, by the convexity of F, one has

$$\mathrm{Gr}(F) - (a, b) \subset T(\mathrm{Gr}(F), (a, b)) = \mathrm{Gr}(DF(a, b)).$$

Then, for any $x \in X$, $F(x) - b \subset DF(a,b)(x-a)$, and so

$$d(0, DF(a,b)(x-a)) \leq d(b, F(x)).$$

On the other hand, the convexity of A implies that

$$d(x-a, T(A,a)) \leq d(x-a, A-a) \leq d(x, A).$$

Hence, for any $x \in X$,

$$d(x-a, T(S,a)) \leq \tau_2(d(0, DF(a,b)(x-a)) + d(x-a, T(A,a))) \leq \tau_2(d(b, F(x)) + d(x, A)).\blacksquare$$

Therefore $\tau_1 \leq \tau_2$ and so $\tau_1 = \tau_2$ is shown. Next we show that $\gamma(F, a, b; A) = \tau_2$. By the definition of τ_2, we have

$$d(x, T(S,a)) \leq \tau_2(d(0, DF(a,b)(x)) + d(x, T(A,a))) \quad \text{for all } x \in X.$$

In the case when $\tau_2 < +\infty$, this implies that $T(A,a) \cap DF(a,b)^{-1}(0) \subset T(S,a)$ and hence (3.12) is seen to hold as the converse inclusion is easily verified by the convexity of F and A. From (3.12) it is straightforward to verify that

$$\gamma(F, a, b; A) = \gamma(DF(a,b), 0, 0; T(A,a)) \quad \text{and} \quad \tau_2 = \tau(DF(a,b), 0, 0; T(A,a)).$$

This and Theorem 3.2 imply that $\tau_2 = \gamma(F, a, b; A)$. In the case when $\tau_2 = +\infty$, suppose to the contrary that $\tau_2 \neq \gamma(F, a, b; A)$. Then, $\gamma(F, a, b; A) < +\infty$. Let $x \in X \setminus T(S,a)$ and $\beta \in (0, 1)$. By Lemma 2.1 there exist $u \in T(S,a)$ and $x^* \in N(T(S,a), u)$ such that

$$\|x^*\| = 1 \quad \text{and} \quad \langle x^*, x - u \rangle \geq \beta \|x - u\|. \tag{3.13}$$

Noting that $N(T(S,a), u) \subset N(T(S,a), 0)$ (because $T(S,a)$ is a closed convex cone), it follows that $x^* \in N(T(S,a), 0) = N(S,a)$ and $\langle x^*, u \rangle = 0$. Take a fixed η in $(\gamma(F, a, b; A), \infty)$. Then there exist $y^* \in \eta B_{Y^*}$, $x_1^* \in D^*F(a,b)(y^*)$ and $x_2^* \in \eta B_{X^*} \cap N(A,a)$ such that $x^* = x_1^* + x_2^*$. Equipping the product space $X \times Y$ with norm $\|(x,y)\|_\eta = \frac{\eta}{1+\eta}\|x\| + \|y\|$ for all $(x,y) \in X \times Y$ and noting that the unit ball of the dual space of $(X \times Y, \|\cdot\|_\eta)$ is $(\frac{\eta+1}{\eta}B_{X^*}) \times B_{Y^*}$, it follows from (2.1) and the convexity of $DF(a,b)$ and A that

$$\frac{1}{\eta}(x_1^*, -y^*) \in N(\mathrm{Gr}(F), (a,b)) \cap ((\frac{\eta+1}{\eta}B_{X^*}) \times B_{Y^*})$$

$$= N(\mathrm{Gr}(DF(a,b)), (0,0)) \cap ((\frac{\eta+1}{\eta}B_{X^*}) \times B_{Y^*})$$

$$= \partial d_{\|\cdot\|_\eta}(\cdot, \mathrm{Gr}(DF(a,b)))(0,0)$$

and

$$\frac{1}{\eta}x_2^* \in B_{X^*} \cap N(A,a) = B_{X^*} \cap N(T(A,a), 0) = \partial d(\cdot, T(A,a))(0).$$

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Therefore,

$$\frac{1}{\eta}\langle x_1^*, x\rangle \leq d_{\|\cdot\|_\eta}((x,0), \mathrm{Gr}(DF(a,b))) \leq d(0, DF(a,b)(x))$$

and $\frac{1}{\eta}\langle x_2^*, x\rangle \leq d(x, T(A,a))$. Noting that $\langle x^*, u\rangle = 0$, it follows from (3.13) that

$$\frac{\beta\|u-x\|}{\eta} \leq d(0, DF(a,b)(x)) + d(x, T(A,a)).$$

Therefore, $\frac{\beta d(x, T(S,a))}{\eta} \leq d(0, DF(a,b)(x)) + d(x, T(A,a))$. Letting $\beta \to 1$, one has

$$d(x, T(S,a)) \leq \eta(d(0, DF(a,b)(x)) + d(x, T(A,a))).$$

This contradicts $\tau_2 = +\infty$. The proof is completed.

Let $\phi : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Consider the special case when $A = X$ and $F(x) = [\phi(x), +\infty)$ for all $x \in X$. In this case, $N(A, x) = \{0\}$ for any $x \in A$. For $a \in \mathrm{dom}(\phi)$, let $\partial^\infty \phi(a)$ denote the singular subdifferential of ϕ at a, namely $\partial^\infty \phi(a) = D^* F(a, \phi(a))(0)$. It is easy to verify from the convexity of ϕ that $\mathrm{dom}(D^* F(a, \phi(a))) \subset R_+$. Thus, noting that

$$D^* F(a, \phi(a))(1) = \partial \phi(a) \text{ and } \partial \phi(a) = \partial \phi(a) + \partial^\infty \phi(a)$$

and adopting the convention that $R_+ \partial \phi(a)$ and $[0, 1]\partial \phi(a)$ are $\{0\}$ if $\partial \phi(a) = \emptyset$, one has

$$\begin{aligned}
D^* F(a, \phi(a))(R) &= D^* F(a, \phi(a))(0) \bigcup D^* F(a, \phi(a))(R_+ \setminus \{0\}) \\
&= D^* F(a, \phi(a))(0) \bigcup R_+ D^* F(a, \phi(a))(1) \\
&= \partial^\infty \phi(a) + R_+ \partial \phi(a)
\end{aligned}$$

and

$$D^* F(a, \phi(a))([-1, 1]) = \partial^\infty \phi(a) + [0, 1]\partial \phi(a).$$

Therefore, our definitions of the BCQ and the strong BCQ for generalized equations are respectively natural generalization of the BCQ and the strong BCQ of a convex inequality system (cf. [17-19, 32]). Thus, Theorems 3.1 and 3.3 extend Theorems 2.2 and 2.3 in [32] from the setting of a convex inequality to that of a convex generalized equation with constraint.

Since the strong BCQ implies the BCQ, the following proposition shows that the converse also holds in some interesting cases.

PROPOSITION 3.1. *Let $a \in S$ and suppose that $N(S, a)$ is a polyhedron in a finite dimensional subspace of X^*. Then (GEC) has the BCQ at a if and only if it has the strong BCQ at a.*

Proof. We need only to show the necessity part. Suppose that (GEC) has the BCQ at a. It suffices to show that there exists $\tau > 0$ such that

$$B_{X^*} \cap N(S,a) \subset \tau(D^*F(a,b)(B_{Y^*}) + B_{X^*} \cap N(A,a)). \qquad (3.14)$$

Let E be a finite dimensional subspace of X^* such that $N(S,a) \subset E$. Let

$$L := N(S,a) \cap -N(S,a),$$

namely L is the largest subspace contained in $N(S,a)$. Take a subspace $L^\perp$ of E such that

$$L \cap L^\perp = \{0\} \text{ and } E = L + L^\perp. \qquad (3.15)$$

Since $N(S,a)$ is a polyhedral cone in E, by [27, Theorem 19.1] there exists a polyhedron cone $C \subset L^\perp$ containing no lines such that

$$N(S,a) = C + L. \qquad (3.16)$$

On the other hand, $\dim(E) < \infty$ and (3.15) imply that there exists $\delta \in (0, +\infty)$ such that $B_{X^*} \cap (C + L) \subset \delta(B_{X^*} \cap C + B_{X^*} \cap L)$. It follows from (3.16) that

$$B_{X^*} \cap N(S,a) \subset \delta(B_{X^*} \cap C + B_{X^*} \cap L) \qquad (3.17)$$

Since L is a finite dimensional space, there exist $l_1, \cdots, l_m \in L$ such that

$$B_{X^*} \cap L \subset \mathrm{co}(l_1, \cdots, l_m). \qquad (3.18)$$

Take $c_1, \cdots, c_n \in C$ such that $C = R^+\mathrm{co}(c_1, \cdots, c_n)$ and $0 \notin \mathrm{co}(c_1, \cdots, c_n)$ (because C is a finite dimensional polyhedron cone containing no lines). Without loss of generality we assume that $B_{X^*} \cap \mathrm{co}(c_1, \cdots, c_n) = \emptyset$. We note that

$$B_{X^*} \cap C \subset \mathrm{co}(0, c_1, \cdots, c_n). \qquad (3.19)$$

By (3.16) and the BCQ assumption, there exist

$$\{y_1^*, \cdots, y_n^*, \tilde{y}_1^*, \cdots, \tilde{y}_m^*\} \subset Y^* \text{ and } \{a_1^*, \cdots, a_n^*, \tilde{a}_1^*, \cdots, \tilde{a}_m^*\} \subset N(A,a)$$

such that

$$c_i \in D^*F(a,b)(y_i^*) + a_i^*, \ 1 \le i \le n \text{ and } l_j \in D^*F(a,b)(\tilde{y}_j^*) + \tilde{a}_j^*, \ 1 \le j \le m.$$

Let $\kappa := \max\limits_{1 \le i \le n, 1 \le j \le m} \|y_i^*\| + \|a_i^*\| + \|\tilde{y}_j^*\| + \|\tilde{a}_j^*\|$. It follows from (3.18) and (3.19) that

$$B_{X^*} \cap C + B_{X^*} \cap L \subset \kappa(D^*F(a,b)(B_{Y^*}) + N(A,a) \cap B_{X^*})$$

This and (3.17) imply that (3.14) holds with $\tau = \delta\kappa$.

COROLLARY 3.1. *Let* $f_1, \cdots, f_n : X \to R \cup \{+\infty\}$ *be proper lower semicontinuous convex functions and consider generalized equation (GEC) with* $A = X$, $Y = R^n$, $b = (b_1, \cdots, b_n) \in R^n$ *and* F *being defined by*

$$F(x) = (f_1(x), \cdots, f_n(x)) + R_+^n \quad \text{for all } x \in X.$$

Suppose that each f_i *is differentiable at* $a \in S$. *Then, for the said generalized equation, the BCQ and the strong BCQ are equivalent at* a.

Proof. In view of Proposition 3.1, it suffices to show that

$$D^*F(a, b)(R^n) = R_+\mathrm{co}\{f_i'(a) : i \in J(a)\}, \tag{3.20}$$

where $J(a) := \{1 \leq i \leq n : f_i(a) = b_i\}$. To do this, we first note that $\mathrm{dom}(D^*F(a, b)) = R_+^n$ (because each convex function f_i is differentiable at a). Let $(r_1, \cdots, r_n) \in R_+^n \setminus \{0\}$ and $x^* \in D^*F(a, b)(r_1, \cdots, r_n)$. Then

$$\langle x^*, x \rangle - \sum_{i=1}^n r_i(f_i(x) + t_i) \leq \langle x^*, a \rangle - \sum_{i=1}^n r_i b_i$$

for any $x \in X$ and $(t_1, \cdots, t_n) \in R_+^n$. Noting that $f_i(a) = b_i$ for any $i \in J(a)$, $f_i(a) < b_i$ for any $i \notin J(a)$ and $a \in \mathrm{int}(\mathrm{dom}(f_i))$ for $1 \leq i \leq n$, it follows that $r_i = 0$ for any $i \notin J(a)$ and

$$\langle x^*, x - a \rangle \leq \sum_{i \in J(a)} r_i f(x) - \sum_{i \in J(a)} r_i f_i(a)$$

for all $x \in X$. This implies that $x^* = \sum_{i \in J(a)} r_i f_i'(a)$. Thus, $x^* \in R_+\mathrm{co}\{f_i'(a) : i \in J(a)\}$. This shows that $D^*F(a, b)(R^n) \subset R_+\mathrm{co}\{f_i'(a) : i \in J(a)\}$. Conversely, let $x^* \in R_+\mathrm{co}\{f_i'(a) : i \in J(a)\}$. Then there exists $(c_1, \cdots, c_n) \in R_+^n$ with $c_i = 0$ for all $i \notin J(a)$ such that $x^* = \sum_{i=1}^n c_i f_i'(a)$. Noting that for each i,

$$\langle c_i f_i'(a), x - a \rangle \leq c_i(f_i(x) + t_i - b_i) \quad \forall x \in X \text{ and } \forall t_i \geq 0.$$

it follows that $x^* \in D^*F(a, b)(c_1, \cdots, c_n)$. This shows that

$$R_+\mathrm{co}\{f_i'(a) : i \in J(a)\} \subset D^*F(a, b)(R^n).$$

Hence (3.20) holds.

4. Calmness of convex multifunctions. Throughout this section, let $M : Y \to 2^X$ be a closed convex multifunction and A be a closed convex subset of X. Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y}) \cap A$. Recall (cf. [8-10] and [15]) that M is said to be calm at $(\bar{y}, \bar{x})$ if there exists a constant $L > 0$ such that

$$d(x, M(\bar{y})) \leq L\|y - \bar{y}\| \quad \text{for all } (y, x) \in \mathrm{Gr}(M) \text{ close to } (\bar{y}, \bar{x}). \tag{4.1}$$

More generally, M is said to be calm at $(\bar{y}, \bar{x})$ over A if there exists a constant $\mathrm{L} > 0$ such that

$$d(x, M(\bar{y}) \cap A) \le L(\|y - \bar{y}\| + d(x, A)) \quad \text{for all } (y, x) \in \mathrm{Gr}(M) \text{ close to } (\bar{y}, \bar{x}). \quad (4.2)$$

Let $\tilde{M} : Y \times X \to 2^X$ be defined by

$$\tilde{M}(y, z) = M(y) \cap (-z + A) \quad \text{for any } (y, z) \in Y \times X$$

and $Y \times X$ be equipped with the norm $\|(y, z)\| = \|y\| + \|z\|$ for any $(y, z) \in Y \times X$. Then, as observed by one of the referees, (4.2) holds if and only if

$$d(x, \tilde{M}(\bar{y}, 0)) \le L\|(y, z) - (\bar{y}, 0)\| \quad \text{for all } (y, z; x) \in \mathrm{Gr}(\tilde{M}) \text{ close to } (\bar{y}, 0; \bar{x}).$$

Hence, M is calm at $(\bar{y}, \bar{x})$ over A if and only if $\tilde{M}$ is calm at $(\bar{y}, 0; \bar{x})$. A more general intersection map have been studied by Klatte and Kummer [16]. Since $d(x, \emptyset) = +\infty$ and $d(x, M(\bar{y})) \le \|x - \bar{x}\|$, it is easy to verify that (4.2) holds if and only if

$$d(x, M(\bar{y}) \cap A) \le L(d(\bar{y}, M^{-1}(x)) + d(x, A)) \quad \text{for all } x \text{ close to } \bar{x} \quad (4.3)$$

Letting $b = \bar{y}$ and $F(x) = M^{-1}(x)$, it follows that (GEC), defined by the data (F, A, b), is metrically subregular at $\bar{x}$ if and only if M is calm at $(\bar{y}, \bar{x})$ over A. Thus, by Theorems 3.1 and 3.3, we have the following results.

THEOREM 4.1. *The following statements are equivalent.*
(i) M is calm at $(\bar{y}, \bar{x})$ over A.
(ii) There exist $\tau, \delta \in (0, +\infty)$ such that for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}(M(\bar{y}) \cap A)$,

$$N(M(\bar{y}) \cap A, u) \cap B_{X^*} \subset \tau(D^* M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) + N(A, u) \cap B_{X^*}).$$

(iii) There exists $\delta \in (0, +\infty)$ such that for all $u \in \mathrm{bd}(M(\bar{y}) \cap A)$ close to $\bar{x}$, the tangent derivative $DM(\bar{y}, u)$ is calm at $(0, 0)$ over $T(A, u)$ with the same constant.

Imitating the notion of the strong metric regularity (cf. [7]), we say that M is strongly calm at $(\bar{y}, \bar{x})$ over A if there exists $L \in [0, +\infty)$ such that

$$\|x - \bar{x}\| \le L(\|y - \bar{y}\| + d(x, A)) \quad \text{for all } (y, x) \in \mathrm{Gr}(M) \text{ close to } (\bar{y}, \bar{x}). \quad (4.4)$$

From the convexity of $M(\bar{y}) \cap A$, it is not difficult to verify that M is strongly calm at $(\bar{y}, \bar{x})$ over A if and only if $M(\bar{y}) \cap A = \{\bar{x}\}$ and M is calm at $(\bar{y}, \bar{x})$ over A. Using Theorem 4.1, we can establish some characterization of the strong calmness.

COROLLARY 4.1. *The following statements are equivalent.*
(i) M is strongly calm at $(\bar{y}, \bar{x})$ over A.
(ii) There exists $L \in [0, +\infty)$ such that

$$\|x - \bar{x}\| \le L(\|y - \bar{y}\| + d(x, A)) \quad \text{for all } (y, x) \in \mathrm{Gr}(M)$$

(iii) The tangent derivative $DM(\bar{y}, \bar{x})$ is strongly calm at $(0,0)$ over $T(A, \bar{x})$.

(iv) $0 \in \text{int}(D^*M^{-1}(\bar{x}, \bar{y})(Y^*) + N(A, \bar{x}))$.

(v) There exists $r > 0$ such that

$$rB_{X^*} \subset D^*M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) + B_{X^*} \cap N(A, \bar{x}).$$

Proof. It is clear that (ii)$\Longrightarrow$(i) and (v)$\Longrightarrow$(iv). We show next that (i)$\Leftrightarrow$(v). By the evident fact $N(\{\bar{x}\}, \bar{x}) = X^*$ and by Theorem 4.1, we need only show that (v)$\Rightarrow M(\bar{y}) \cap A = \{\bar{x}\}$. Take an arbitrary $x \in M(\bar{y}) \cap A$ and $x^* \in B_{X^*}$ such that $\|x - \bar{x}\| = \langle x^*, x - \bar{x} \rangle$. By (v), there exist $y^* \in B_{Y^*}$, $x_1^* \in D^*M^{-1}(\bar{x}, \bar{y})(y^*)$ and $x_2^* \in N(A, \bar{x}) \cap B_{X^*}$ such that $rx^* = x_1^* + x_2^*$. Hence

$$r\|x - \bar{x}\| = \langle x_1^*, x - \bar{x} \rangle + \langle x_2^*, x - \bar{x} \rangle.$$

Noting that $\langle x_1^*, x - \bar{x} \rangle \leq \langle y^*, \bar{y} - \bar{y} \rangle = 0$ and $\langle x_2^*, x - \bar{x} \rangle \leq 0$, it follows that $r\|x - \bar{x}\| \leq 0$ for any $x \in M(\bar{y}) \cap A$. This shows that $M(\bar{y}) \cap A = \{\bar{x}\}$; thus (i)$\Leftrightarrow$(v).

Noting that $D^*M^{-1}(\bar{x}, \bar{y}) = D^*(DM(\bar{y}, \bar{x}))^{-1}(0, 0)$, we have (iii)$\Leftrightarrow$(v) by (i)$\Leftrightarrow$(v), applied to $DM(\bar{y}, \bar{x})$ in place of M.

Suppose that (i) holds. Then there exists $L \in [0, +\infty)$ such that (4.4) holds. Let (y, x) be an arbitrary element in $\text{Gr}(M)$ and $t \in (0, 1)$ be small enough such that $(ty + (1 - t)\bar{y}, tx + (1 - t)\bar{x})$ close enough to $(\bar{y}, \bar{x})$. By (4.4) and the convexity of M, one has

$$\|tx + (1 - t)\bar{x} - \bar{x}\| \leq L(\|ty + (1 - t)\bar{y} - \bar{y}\| + d(tx + (1 - t)\bar{x}, A)).$$

It follows from the convexity of A and $\bar{x} \in A$ that $\|x - \bar{x}\| \leq L(\|y - \bar{y}\| + d(x, A))$. This shows that (i)$\Longrightarrow$(ii).

It remains to show that (iv)$\Rightarrow$(v). Suppose that (iv) holds. Since $N(A, x)$ and $D^*M^{-1}(x, \bar{y})(Y^*)$ are cones,

$$X^* = D^*M^{-1}(\bar{x}, \bar{y})(Y^*) + N(A, \bar{x}) = \bigcup_{n=1}^{\infty} (D^*M^{-1}(\bar{x}, \bar{y})(nB_{Y^*}) + N(A, \bar{x}) \cap nB_{X^*}).$$

Noting that, by the Alaoglu theorem, each set $D^*M^{-1}(\bar{x}, \bar{y})(nB_{Y^*}) + N(A, \bar{x}) \cap nB_{X^*}$ is weak* closed, it follows from the well-known Baire category theorem and (iv) that

$$0 \in \text{int}(D^*M^{-1}(\bar{x}, \bar{y})(B_{Y^*}) + N(A, \bar{x}) \cap B_{X^*}).$$

Hence there exists $r > 0$ such that (v) holds. The proof is completed.

Remark. In a recent paper [8] Henrion and Jourani considered the calmness of the convex multifunction M_0 of the following type

$$M_0(y) = \{x \in C : f(x) \leq y\} \quad \text{for all } y \in R, \tag{4.5}$$

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where C is a closed convex subset of X and $f : X \to R \cup \{+\infty\}$ is a proper lower semicontinuous convex function. In particular, as a main result, they established the following result .

Theorem HJ ([8, Theorem 3.3]). Let M_0 be defined by (4.5). Then M_0 is calm at $(0, \bar{x}) \in \mathrm{Gr}(M_0)$ if one of the following conditions is satisfied:

(C1) $f(\bar{x}) < 0$.

(C2) $\mathrm{bd}\partial f(\bar{x}) \cap \mathrm{bd}N(C, \bar{x}) \neq \partial f(\bar{x}) \cap N(C, \bar{x})$.

(C3) $\mathrm{bd}\partial f(\bar{x}) \cap \mathrm{bd}N(C, \bar{x}) = \emptyset$ and (CD*) (see [8] for the definition of condition (CD*)).

As observed by Henrion and Jourani [8], (C3)$\Longrightarrow$(C2) or (C1) and

$$(C2) \Longrightarrow \mathrm{int}\partial f(\bar{x}) \cap -N(C, \bar{x}) \neq \emptyset \text{ or } \partial f(\bar{x}) \cap -\mathrm{int}N(C, \bar{x}) \neq \emptyset.$$

Hence (C2) $\Longrightarrow 0 \in \mathrm{int}(\partial f(\bar{x}) + N(C, \bar{x}))$. Considering that (C1)$\Longrightarrow$the calmness of M_0 at $(0, \bar{x})$ is an immediate consequence of the Robinson-Ursescu theorem (cf. [25,29]), the main part of Theorem HJ can be rewritten as follows.

Theorem HJ$'$. M_0 is calm at $(0, \bar{x}) \in \mathrm{Gr}(M_0)$ if

$$f(\bar{x}) = 0 \text{ and } 0 \in \mathrm{int}(\partial f(\bar{x}) + N(C, \bar{x})). \tag{4.6}$$

Let $A = C$, $Y = R$ and $M(y) = \{x \in X : f(x) \leq y\}$ for all $y \in Y$. It is clear that M_0 is calm at $(0, \bar{x})$ if M is calm at $(0, \bar{x})$ over A. Since

$$\partial f(\bar{x}) = D^* M^{-1}(\bar{x}, f(\bar{x}))(1) \subset D^* M^{-1}(\bar{x}, f(\bar{x}))(Y^*),$$

(4.6) is stronger than (v) in Corollary 4.1. Hence Corollary 4.1 improves Theorem HJ$'$. Similarly, one can see that Corollary 4.1 improve [8, Theorem 4.3] when $0 \notin \mathrm{int}(C-D)$. As in the beginning of the proof of [8, Theorem 4.3], it is an immediate consequence of the Robinson-Ursescu theorem when $0 \notin \mathrm{int}(C - D)$.

The calmness modulus of M at $(\bar{y}, \bar{x})$ is denoted by $\eta(M; \bar{y}, \bar{x})$ and is defined by

$$\eta(M; \bar{y}, \bar{x}) := \inf\{L \in (0, +\infty) : (4.1) \text{ holds}\}.$$

As applications of Theorems 3.1 and 3.2, we establish formulas representing $\eta(M; \bar{y}, \bar{x})$.

THEOREM 4.2. $\eta(M; \bar{y}, \bar{x}) = \limsup\limits_{u \overset{\mathrm{bd}(M(\bar{y}))}{\longrightarrow} \bar{x}} \|D^* M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^-$

Proof. Since (4.2)$\Leftrightarrow$(4.3), $\eta(M; \bar{y}, \bar{x}) = \tau(M^{-1}, \bar{x}, \bar{y}; X)$. By (3.6), it suffices to show that

$$\gamma(M^{-1}, u, \bar{y}; X) = \|D^* M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^- \quad \forall u \in M(\bar{y}). \tag{4.7}$$

Let $u \in M(\bar{y})$ and $\tau > \gamma(M^{-1}, u, \bar{y}; X)$. Noting that $N(X, u) = \{0\}$, one has

$$N(M(\bar{y}), u) \cap B_{X^*} \subset D^* M^{-1}(u, \bar{y})(\tau B_{Y^*}).$$

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Hence, for any $x^* \in N(M(\bar{y}), u) \cap B_{X^*}$ there exists $y^* \in B_{Y^*}$ such that $x^* \in D^*M^{-1}(u, \bar{y})(\tau y^*)$, that is, $-\tau y^* \in D^*M(\bar{y}, u)(-x^*)$. It follows that

$$\|D^*M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^- \leq \tau.$$

Therefore, $\|D^*M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^- \leq \gamma(M^{-1}, u, \bar{y}; X)$. To prove the converse inequality, let $\tau > \|D^*M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^-$ and $x^* \in N(M(\bar{y}), u) \cap B_{X^*}$. Then, there exists $y^* \in D^*M(\bar{y}, u)(-x^*)$ such that $\|y^*\| < \tau$; this implies that $x^* \in D^*M^{-1}(u, \bar{y})(\tau B_{Y^*})$. Hence, $N(M(\bar{y}), u) \cap B_{X^*} \subset D^*M^{-1}(u, \bar{y})(\tau B_{Y^*})$. It follows that $\gamma(M^{-1}, u, \bar{y}; X) \leq \tau$. Therefore,

$$\gamma(M^{-1}, u, \bar{y}; X) \leq \|D^*M(\bar{y}, u)|_{-N(M(\bar{y}), u)}\|^-.$$

This shows that (4.7) holds.

Remark. In contrast to formula (1.4) of the modulus of the metric regularity, $\eta(M; \bar{y}, \bar{x})$ is not necessarily equivalent to $\|D^*M(\bar{y}, \bar{x})|_{-N(M(\bar{y}), \bar{x})}\|^-$ even when $Y = R^2$ and $X = R$ (cf. [30, Example 2]). Nevertheless, the following theorem shows an interesting case for which the equality holds.

THEOREM 4.3. *Suppose that there exist a cone C and a neighborhood V of $\bar{x}$ such that $M(\bar{y}) \cap V = (\bar{x} + C) \cap V$. Then $\eta(M; \bar{y}, \bar{x}) = \|D^*M(\bar{y}, \bar{x})|_{-N(M(\bar{y}), \bar{x})}\|^-$.*

The proof of Theorem 4.3 is similar to that of Theorem 4.2 but using Theorem 3.2 in place of Theorem 3.1.

5. Recession core and global metric subregularity. Let K be a closed convex subset of X. Recall that $e \in K$ is called an extreme point of K if $x_1 = x_2$ whenever $e = tx_1 + (1 - t)x_2$ with $x_1, x_2 \in K$ and $t \in (0, 1)$. We denote by $\text{ext}(K)$ the set of all extreme points of K (usually $\text{ext}(K)$ is called the extreme boundary of K). Let K^∞ denote the recession cone of K, that is,

$$K^\infty := \{h \in X : K + th \subset K \text{ for all } t \geq 0\}.$$

It is known that K^∞ is a closed convex cone, and

$$K^\infty = \{h \in X : x + R_+ h \subset K \text{ for some } x \in K\}$$
$$= \{h \in X : \exists x_n \in K \text{ and } \exists t_n > 0 \text{ such that } t_n \to 0 \text{ and } t_n x_n \to h\}.$$

Clearly, $K + K^\infty = K$. It is well known that if K is a closed convex subset of R^n containing no lines then $K = \text{co}(\text{ext}(K)) + K^\infty$. As a generalization of $\text{co}(\text{ext}(K))$, the authors [24] introduced the concept of recession property: a convex subset A of K is said to have the recession property if $K = A + K^\infty$. Simplifying the recession property, now we can give a generalization of $\text{ext}(K)$: a subset C of K is said to be a recession core of K if

$$K = \text{co}(C) + K^\infty. \tag{5.1}$$

Thus, C is a recession core of K if and only if $co(C)$ is a subset of K with recession property. Let "$\leq_{K^\infty}$" denote the order induced by the cone K^∞, that is, $x_1 \leq_{K^\infty} x_2$ if and only if $x_2 - x_1 \in K^\infty$. Let A be a subset of X. We say that $a \in A$ is a minimal element of A with respect to "$\leq_{K^\infty}$" if $a \leq_{K^\infty} x$ whenever $x \in A$ and $x \leq_{K^\infty} a$. We denote by $\text{Min}(A, K^\infty)$ the set of all minimal elements of A. Let

$$\text{Lin}(K) := K^\infty \cap -K^\infty.$$

Then

$$\text{Lin}(K) = \{h \in X : K + Rh \subset K\} = \{h \in X : x + Rh \subset K \text{ for some } x \in K\}.$$

Moreover, $K = K + \text{Lin}(K)$ and $L \subset \text{Lin}(K)$ whenever L is a subspace of X such that $K = K + L$. Therefore, K contains no lines if and only if $\text{Lin}(K) = \{0\}$.

PROPOSITION 5.1. *Let X be a reflexive Banach space and K a closed convex nonempty subset of X. Suppose that there exists a closed convex bounded set Θ such that*

$$\text{Lin}(K) \cap \Theta = \emptyset \text{ and } K^\infty = \text{Lin}(K) + R_+\Theta. \tag{5.2}$$

Then

$$K = \text{Min}(K, K^\infty) + K^\infty. \tag{5.3}$$

In particular, $\text{Min}(K, K^\infty)$ is a recession core of K.

Proof. By the reflexivity of X, the bounded closed convex set Θ is weakly compact. Letting $K_0 := R_+\Theta$, it follows that K_0 is a closed convex pointed cone. Let x be an arbitrary point in K. We claim that $K \cap (x - K_0)$ is bounded. If this is not the case, then there exist a sequence $\{s_n\}$ in R_+ and a sequence $\{\theta_n\}$ in Θ such that $s_n \to \infty$ and $x - s_n\theta_n \in K$ for all n. Thus, for any $t > 0$,

$$x - t\theta_n = (1 - \frac{t}{s_n})x + \frac{t}{s_n}(x - s_n\theta_n) \in K \text{ for all } n \text{ large enough.}$$

By the weak compactness of Θ, without loss of generality we can assume that $\{\theta_n\}$ converges weakly to some $\theta \in \Theta$. Therefore, $x - t\theta \in K$ for any $t \geq 0$. This implies that $-\theta \in K^\infty$. On the other hand, by the second equality in (5.2), one has $\theta \in K^\infty$, and so $\theta \in \text{Lin}(K)$. This contradicts the first equality in (5.2) and therefore $K \cap (x - K_0)$ must be bounded (and hence weakly compact). It follows from [14, Corollary 3.1.16]) that $\text{Min}(K \cap (x - K_0), K_0) \neq \emptyset$. Take $x' \in \text{Min}(K \cap (x - K_0), K_0)$. Then $x' \in \text{Min}(K, K_0)$ and $x \in x' + K_0$. Hence $K \subset \text{Min}(K, K_0) + K_0$. Now to show (5.3), it suffices to show that $\text{Min}(K, K_0) \subset \text{Min}(K, K^\infty)$. Let $z \in \text{Min}(K, K_0)$ and $y \in K$ with $y \leq_{K^\infty} z$. Then $z - y \in K^\infty$. By the second equality of (5.2), there exists $e \in \text{Lin}(K)$ such that $z - y - e \in K_0$, that is, $y + e \leq_{K_0} z$. Noting that $y + e \in K$, one has that $y + e - z \in K_0$ and hence $y - z \in \text{Lin}(K) + K_0 = K^\infty$. This shows that

$z \in \text{Min}(K, K^\infty)$. The proof is completed.

In the case when X is finite dimensional, the assumption made in (5.2) automatically holds (by $K = \text{Lin}(K) + C$ and Klee's Theorem (cf. [14]), where C is a closed convex pointed cone). The following example shows that the reflexivity of X cannot be removed in Proposition 5.1.

Example. Let $X = l^1$ and $K = \{x = (t_1, t_2, \cdots) \in l^1 : t_n \geq -n, \quad \forall n\}$. It is easy to verify that $\text{Lin}(K) = \{0\}$ and $K^\infty = \{x = (t_1, t_2, \cdots) \in l^1 : t_n \geq 0, \quad \forall n\}$. Let $\Theta := \{x = (t_1, t_2, \cdots) \in K^\infty : \sum_{n=1}^{\infty} t_n = 1\}$. Then, Θ is a bounded closed convex set, $\text{Lin}(K) \cap \Theta = \emptyset$ and $K^\infty = R^+ \Theta$. But $\text{Min}(K, K^\infty) = \emptyset$, and hence $K \neq \text{Min}(K, K^\infty) + K^\infty$. Indeed, let $x = (t_1, t_2, \cdots)$ be any point in K. Noting that $\sum_{n=1}^{\infty} |t_n| < \infty$, there exists a natural number n_0 such that $|t_n| < n_0$ for all $n \geq n_0$. Take $x_0 = (s_1, s_2, \cdots)$ to satisfy $s_n = t_n$ for any $n \neq n_0$ and $s_{n_0} = -n_0$. It is clear that $x_0 \in K \setminus \{x\}$ and $x - x_0 \in K^\infty$. It follows that $x \notin \text{Min}(K, K^\infty)$. This shows that $\text{Min}(K, K^\infty) = \emptyset$.

It is clear that K has no extreme points if K contains lines. This motivates us to introduce a new concept of what we shall refer to as "generalized extreme points". Let X be a Hilbert space. For a closed convex subset K of X, let

$$\text{Lin}(K)^\perp := \{x \in X : \langle x, y \rangle = 0, \quad \forall y \in \text{Lin}(K)\}.$$

We say that e is a generalized extreme point of K if $e \in K \cap \text{Lin}(K)^\perp$ and

$$x_1, x_2 \in K \text{ and } e = \frac{x_1 + x_2}{2} \implies x_1 - x_2 \in \text{Lin}(K). \tag{5.4}$$

We denote by $\text{ext}_E(K)$ the set of all generalized extreme points of K. Clearly, $\text{ext}_E(K) = \text{ext}(K)$ if K contains no lines (i.e., $\text{Lin}(K) = \{0\}$). Moreover, one has that

$$\text{ext}_E(K) \subset \text{Min}(K, K^\infty). \tag{5.5}$$

To see this, let $e \in \text{ext}_E(K)$ and $x \in K$ with $x \leq_{K^\infty} e$. Then $e - x \in K^\infty$ and hence $2e - x = e + (e - x) \in K$. Since $e = \frac{x + (2e - x)}{2}$, it follows that $2(e - x) \in \text{Lin}(K)$, which means $x - e \in \text{Lin}(K) \subset K^\infty$. Hence $e \leq_{K^\infty} x$. This shows that $e \in \text{Min}(K, K^\infty)$. Thus (5.5) is true.

PROPOSITION 5.2. *Let X be a Hilbert space and K a closed convex subset of X. Then*

$$\text{ext}_E(K) = \text{ext}(K \cap \text{Lin}(K)^\perp). \tag{5.6}$$

Proof. By definition it is clear that $\text{ext}_E(K) \subset \text{ext}(K \cap \text{Lin}(K)^\perp)$. Conversely, let $e \in \text{ext}(K \cap \text{Lin}(K)^\perp)$ and $x_1, x_2 \in K$ satisfy $e = \frac{x_1 + x_2}{2}$. Noting that for each

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$x \in X$ there exists a unique pair $(u, v) \in \mathrm{Lin}(K) \times \mathrm{Lin}(K)^\perp$ such that $x = u + v$. Take $(u_1, v_1), (u_2, v_2) \in \mathrm{Lin}(K) \times \mathrm{Lin}(K)^\perp$ such that $x_1 = u_1 + v_1$ and $x_2 = u_2 + v_2$. Then $e = \frac{u_1 + u_2}{2} + \frac{v_1 + v_2}{2}$. It follows from $e \in \mathrm{ext}(K \cap \mathrm{Lin}(K)^\perp)$ that $u_1 + u_2 = 0$ and $e = v_1 = v_2$. Thus, $x_1 - x_2 = u_1 - u_2 \in \mathrm{Lin}(K)$ and hence $e \in \mathrm{ext}_E(K)$. This shows that $\mathrm{ext}_E(K) \supset \mathrm{ext}(K \cap \mathrm{Lin}(K)^\perp)$, and (5.6) is proved.

PROPOSITION 5.3. *Let K be a closed convex subset of a Hilbert space X and $\prod$ be the project operator to $\mathrm{Lin}(K)^\perp$. Suppose that C is a recession core of K. Then $\mathrm{ext}_E(K) \subset \prod(C)$.*

Proof. Let $e \in \mathrm{ext}_E(K)$. Then, by (5.1) there exist $x_1, \cdots, x_n \in C$, $t_1, \cdots, t_n \in [0, 1]$ with $\sum\limits_{i=1}^{n} t_i = 1$ and $h \in K^\infty$ such that $e = \sum\limits_{i=1}^{n} t_i x_i + h$. Then,

$$e = \frac{(\sum\limits_{i=1}^{n} t_i x_i + \frac{1}{2}h) + (\sum\limits_{i=1}^{n} t_i x_i + \frac{3}{2}h)}{2}.$$

Thus, by (5.4), one has $h \in \mathrm{Lin}(K)$. It follows from the linearity of $\prod$ that

$$e = \prod(e) = \prod(\sum_{i=1}^{n} t_i x_i + h) = \sum_{i=1}^{n} t_i \prod(x_i) \in \prod(C).$$

This shows that $\mathrm{ext}_E(K) \subset \prod(C)$.

The following proposition shows that $\mathrm{ext}_E(K)$ is a recession core of K when X is finite dimensional.

PROPOSITION 5.4. *Let K be a closed convex nonempty subset of R^n. Then*

$$K = \mathrm{co}(\mathrm{ext}_E(K)) + K^\infty. \tag{5.7}$$

Proof. Let $h \in R^n$ be such that $x + Rh \subset K \cap \mathrm{Lin}(K)^\perp$ for some $x \in R^n$. Then $h \in \mathrm{Lin}(K)$, and hence $\langle x + th, h \rangle = 0$ for all $t \in R$. It follows that $h = 0$. Therefore $K \cap \mathrm{Lin}(K)^\perp$ is a closed convex subset containing no lines. It follows from [27, Theorem 18.5] that

$$K \cap \mathrm{Lin}(K)^\perp = \mathrm{co}(\mathrm{ext}(K \cap \mathrm{Lin}(K)^\perp)) + (K \cap \mathrm{Lin}(K)^\perp)^\infty.$$

This and Proposition 5.2 imply that $K \cap \mathrm{Lin}(K)^\perp \subset \mathrm{co}(\mathrm{ext}_E(K)) + K^\infty$. Noting that $K^\infty + \mathrm{Lin}(K) = K^\infty$, one then has

$$K \cap \mathrm{Lin}(K)^\perp + \mathrm{Lin}(K) \subset \mathrm{co}(\mathrm{ext}_E(K)) + K^\infty. \tag{5.8}$$

Let $x \in K$ and take $x_1 \in \mathrm{Lin}(K)$ and $x_2 \in \mathrm{Lin}(K)^\perp$ such that $x = x_1 + x_2$. Hence $x_2 \in x + \mathrm{Lin}(K) \subset K$ and so $x_2 \in K \cap \mathrm{Lin}(K)^\perp$. Therefore, $K \subset \mathrm{Lin}(K) + K \cap \mathrm{Lin}(K)^\perp$. It follows from (5.8) that

$$K \subset \mathrm{co}(\mathrm{ext}_E(K)) + K^\infty.$$

Thus (5.7) holds as the converse inclusion is obvious.

Remark. Proposition 5.4 shows that $\text{ext}_E(K)$ is a recession core of K, and has the minimality property "up to $\text{Lin}(K)$" in the sense as indicated in Proposition 5.3. In particular, if K contains no lines then $\text{ext}_E(K) = \text{ext}(K)$ is the least recession core of K.

In terms of recession cores and the BCQs, we now study the global metric subregularity of generalized equation (GEC). Hoffman, in his pioneering work, proved that (GEC) has an error bound (or equivalently, is globally metrically subregular) if $A = X = R^n$ and $F(x) := Qx + R_+^n$ for all $x \in R^n$, where Q is a $m \times n$ matrix. The research on error bounds, especially for inequality systems, has attracted the interest of many researchers and there are a vast number of publications reporting progress in this area. For more details, see [17,18,20,23,24,30,31] and a special issue of Mathematical Programming (Vol.88, No.2, 2000). In what follows, we assume that X, Y are general Banach spaces (except explicitly stated otherwise), F is a closed convex multifunction from X to Y and that A is a closed convex subset of X. In the case when $A = X$, while the equivalence of (iv) with (v) in the following result is [24, Theorem 3.1], we can now sharpen the result by considering recession cores of S.

THEOREM 5.1. *Let C be a recession core of the solution set S of (GEC) and $\tau \in [0, +\infty)$. Then the following statements are equivalent.*
(i) (GEC) has the strong BCQ at each $x \in C$ with the constant τ.
(ii) (GEC) has the strong BCQ at each $x \in S$ with the constant τ.
(iii) (GEC) is metrically subregular at each point in C with the constant τ.
(iv) (GEC) is metrically subregular at each point in S with the constant τ.
(v) (GEC) is globally metrically subregular with the constant τ.

Proof. (i)$\Rightarrow$(ii) Let $x \in S$. Since C is a recession cone of S, there exist $x_1, \cdots, x_n \in C$, $t_1, \cdots, t_n \in [0, +\infty)$ and $e \in S^\infty$ such that

$$\sum_{i=1}^n t_i = 1 \ \text{ and } \ x = \sum_{i=1}^n t_i x_i + e. \tag{5.9}$$

Let $x^* \in N(S, x) \cap B_{X^*}$. Then $\langle x^*, \sum_{i=1}^n t_i x_i + e \rangle = \max\{\langle x^*, z \rangle : z \in S\}$. Noting that $\sum_{i=1}^n t_i x_i + R_+ e \subset S$, it follows that

$$\langle x^*, e \rangle = 0 \ \text{ and } \ \langle x^*, \sum_{i=1}^n t_i x_i \rangle = \max\{\langle x^*, z \rangle : z \in S\}.$$

This implies that for each integer $i \in [1, \ n]$,

$$\langle x^*, x \rangle = \langle x^*, x_i \rangle = \max\{\langle x^*, z \rangle : z \in S\},$$

and hence $x^* \in N(S, x_i) \cap B_{X^*}$. By (i), one has

$$x^* \in \tau(D^* F(x_i, b)(B_{Y^*}) + B_{X^*} \cap N(A, x_i)), i = 1, \cdots, n.$$

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It follows from Lemma 3.1 that $x^* \in \tau(D^*F(x, b)(B_{Y^*}) + B_{X^*} \cap N(A, x))$. Therefore,

$$N(S, x) \cap B_{X^*} \subset \tau(D^*F(x, b)(B_{Y^*}) + B_{X^*} \cap N(A, x)).$$

This shows that (ii) holds.

(ii)$\Rightarrow$(i), (iv)$\Rightarrow$(iii) and (v)$\Rightarrow$(iv) are trivial. (iii)$\Rightarrow$(i) and (ii)$\Rightarrow$(iv) are consequences of formula (3.6) in Remark 3.1. The proof of (iv)$\Rightarrow$(v) is similar to that of [24, Theorem 3.1]. The proof is completed.

In the special case when $A = X$, $Y = R$, $F(x) = [f(x), +\infty)$ for all $x \in X$ and $b = \inf\{f(x) : x \in X\}$ with f being a proper lower semicontinuous convex function from X to $R \cup \{+\infty\}$, Burke and Deng [3, Theorem 2.3] proved that (GEC) is globally τ-metrically subregular if and only if

$$N(S, z) \cap B_{X^*} \subset \mathrm{cl}^*(\partial f(z)) \text{ for all } z \in S.$$

Since $\mathrm{ext}_E(S)$ is a recession core of S if $X = R^n$, the following corollary is a consequence of Theorem 5.1.

COROLLARY 5.1. *Let $X = R^n$. Then (GEC) is globally metrically subregular if and only there exists $\tau \in (0, +\infty)$ such that (GEC) has the strong BCQ at each generalized extreme point of S with the constant τ.*

Similar to the proof of the equivalent relation (i)$\Leftrightarrow$(ii) in Theorem 5.1, one can prove the following result.

PROPOSITION 5.5. *Let C be a recession core of S. Then (GEC) has the BCQ at each point in C if and only if (GEC) has the BCQ at each point in S.*

As in the finite dimensional case, let us say that a subset P of X is a polyhedron if there exist $x_n^*, \cdots, x_n^* \in X^*$ and $c_1, \cdots, c_n \in R$ such that

$$P = \{x \in X : \langle x_i^*, x \rangle \le c_i, \ i = 1, \cdots, n\}.$$

It is known that

$$N(P, x) = \left\{ \sum_{i \in I(x)} t_i x_i^* : t_i \ge 0, \ i \in I(x) \right\} \quad \forall x \in P, \tag{5.10}$$

where $I(x) := \{1 \le i \le n : \langle x_i^*, x \rangle = c_i\}$. We say that a multifunction $F : X \to 2^Y$ is polyhedral if its graph is a polyhedron in $X \times Y$. If F is polyhedral, it is easy to verify from (5.10) that

$$N(F^{-1}(b), x) = D^*F(x, b)(Y^*) \quad \forall x \in F^{-1}(b). \tag{5.11}$$

THEOREM 5.2. *Let C be a recession core of the solution set S of (GEC). Suppose that S is a polyhedron in X. Then (GEC) is globally metrically subregular if and only if (GEC) has the BCQ at each point in C.*

Proof. By Theorem 5.1, it suffices to prove the sufficiency. Suppose that (GEC) has the BCQ at each point in C. Then, (GEC) has the BCQ at each point of S (by Proposition 5.5). Since S is a polyhedron, there exist $x_n^*, \cdots, x_n^*$ and $c_1, \cdots, c_n \in R$ such that

$$S = \{x \in X : \langle x_i^*, x \rangle \leq c_i, \ i = 1, \cdots, n\}.$$

Let $X_1 := \{x \in X : \langle x_i^*, x \rangle = 0, \ i = 1, \cdots, n\}$. Then X_1 is a closed subspace of X with finite codimension. Thus, there exists a finite dimensional subspace X_2 of X such that $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$. Let

$$P := \{z \in X_2 : \langle x_i^*, z \rangle \leq c_i, \ i = 1, \cdots, n\}.$$

It is easy to verify that $S = P + X_1$ and P is a polyhedron containing no lines in X_2. By [27, Theorems 18.5 and 19.1], $P = \text{co}(\text{ext}(P)) + P^\infty$. Hence $S = \text{co}(\text{ext}(P)) + P^\infty + X_1$. Noting that $P^\infty + X_1 \subset (S)^\infty$, it follows that $\text{ext}(P)$ is a recession core of S. Let $e \in \text{ext}(P)$. Then, by (5.10), $N(S, e)$ is a polyhedron in a finite dimensional subspace of X^*. It follows from Proposition 3.1 that there exists $\tau_e \in (0, +\infty)$ such that (GEC) has the strong BCQ with the constant τ_e. Do this for each e in $\text{ext}(P)$ and let $\tau := \max\{\tau_e : e \in \text{ext}(P)\}$. Then $\tau < +\infty$ because $\text{ext}(P)$ is a finite set (cf. [27, Theorem 19.1]). Hence (GEC) has the strong BCQ at each point of $\text{ext}(P)$ with the constant τ. Since $\text{ext}(P)$ is a recession core of S, it follows from Theorem 5.1 that (GEC) is globally metrically subregular. The proof is completed.

In view of the proof of Theorem 5.2, one sees that any polyhedron in a Banach space has a recession core consisting of finitely many elements.

Robinson [26] studied the continuity properties of polyhedral multifunctions. In particular, under the finite-dimension assumption, he [26, Corollary] proved that if the graph of F is the union of finitely many polyhedra and $b \in F(X)$ then there exists $\varepsilon, \tau \in [0, +\infty)$ such that

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) \ \text{ for all } x \in X \text{with } d(b, F(x)) < \varepsilon.$$

This result can be regarded as a generalization of Hoffman's classical error bound theorem. In the setting of Theorem 5.2, F is not required to be polyhedral but merely the solution set S is required to be polyhedral. When F is a convex polyhedral multifunction and $A = X$, (5.11) implies that generalized equation (GEC) has the BCQ at each $x \in S = F^{-1}(b)$. Hence, in this case, Theorem 5.2 improves the above Robinson's result. But, in the nonconvex case, one cannot use Theorem 5.2 to deduce Robinson's result.

Remark. Let $M : Y \to 2^X$ be a closed convex multifunction and with $(\bar{y}, \bar{x}) \in \text{Gr}(M)$. We say that M is globally calm at $\bar{y}$ over A if there exists $\tau \in (0, +\infty)$ such

that

$$d(x, M(\bar{y}) \cap A) \leq \tau(\|y - \bar{y}\| + d(x, A)) \text{ for all } (x, y) \in \text{Gr}(M).$$

Let C be a recession core of $M(\bar{y}) \cap A$. Theorems 5.1 implies that M is globally calm at $\bar{y}$ over A if and only if there exists $\tau \in (0, +\infty)$ such that

$$N(M(\bar{y}) \cap A, u) \cap B_{X^*} \subset \tau(D^* M^{-1}(u, \bar{y})(B_{Y^*}) + N(A, u) \cap B_{X^*}) \quad \forall u \in C.$$

In the case when $M(\bar{y}) \cap A$ is a polyhedron, Theorem 5.2 implies that M is globally calm at $\bar{y}$ over A if and only if

$$N(M(\bar{y}) \cap A, u) = D^* M^{-1}(u, \bar{y})(Y^*) + N(A, u) \quad \forall u \in C.$$

To end this paper, we provide a procedure to find the generalized extreme points of a polyhedron in a finite dimensional space. Let $a_1, \cdots, a_m \in R^n$, $c_1, \cdots, c_m \in R$ and let P denote the polyhedron determined by a_i and c_i ($i = 1, \cdots, m$), that is,

$$P = \{x \in R^n : \langle a_i, x \rangle \leq c_i, \ i = 1, \cdots, m\}.$$

For convenience, let $I := \{1, \cdots, m\}$ and $I(x) = \{i \in I : \langle a_i, x \rangle = c_i\}$ for $x \in P$. Let $\mathcal{M}(I)$ denote the family of all subsets D of I with the property that $\{a_i : i \in D\}$ is a maximal linearly independent subset of $\{a_i : i \in I\}$. Thus elements of $\mathcal{M}(I)$ can be obtained by the Gram-Schmidt Process. For each $D \in \mathcal{M}(I)$, the following linear equation system

$$\sum_{j \in D} \langle a_i, a_j \rangle t_j = c_i, \quad \forall i \in D$$

has a unique solution which will be denoted by $(\bar{t}_j)_{j \in D}$; we shall also write e_D for $\sum_{j \in D} \bar{t}_j a_j$. Let

$$\mathcal{E}(I) = \{D \in \mathcal{M}(I) : \langle a_i, e_D \rangle \leq c_i, \ i \in I \setminus D\}.$$

THEOREM 5.3. $\text{ext}_E(P) = \{e_D : D \in \mathcal{E}(I)\}$.
Proof. Note that $\text{Lin}(P) = \{x \in R^n : \langle a_i, x \rangle = 0, \ i \in I\}$. Hence,

$$\text{Lin}(P)^\perp = \text{span}\{a_i : i \in I\} = \text{span}\{a_i : i \in D\} \quad \forall D \in \mathcal{M}(I), \tag{5.12}$$

where $\text{span}A$ denotes the linear hull of A. Let $e \in \text{ext}_E(P)$ and pick a $D_0 \subset I(e)$ such that $\{a_i : i \in D_0\}$ is a maximal linearly independent subset of $\{a_i : i \in I(e)\}$. We claim that $D_0 \in \mathcal{M}(I)$. Indeed, if this is not the case, then $\text{span}\{a_i : i \in I(e)\}$ is a proper subspace of $\text{span}\{a_i : i \in I\}$. It follows from the first equality of (5.12) that there exists $h \in \text{Lin}(P)^\perp \setminus \{0\}$ such that $\langle a_i, h \rangle = 0$ for all $i \in I(e)$. Since $\langle a_i, e \rangle < c_i$ for all $i \in I \setminus I(e)$, there exists $\varepsilon > 0$ small enough such that $e \pm \varepsilon h \in P$. Since $e = \frac{e + \varepsilon h + (e - \varepsilon h)}{2}$, it follows from (5.4) that $2\varepsilon h \in \text{Lin}(P)$. This contradicts

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$h \in \text{Lin}(P)^{\perp} \setminus \{0\}$. Hence $D_0 \in \mathcal{M}(I)$. Noting that $e \in \text{Lin}(P)^{\perp}$ (by Proposition 5.2), it follows from (5.12) that there exists $(\bar{t}_j)_{j \in D_0} \in R^{|D_0|}$ such that $e = \sum_{j \in D_0} \bar{t}_j a_j$, where $|D_0|$ denotes the number of elements of D_0. It follows from $e \in P$ and $D_0 \subset I(e)$ that $D_0 \in \mathcal{E}(I)$ and $e = e_{D_0}$. Therefore, $\text{ext}_E(P) \subset \{e_D : D \in \mathcal{E}(I)\}$. It remains to show that $\{e_D : D \in \mathcal{E}(I)\} \subset \text{ext}_E(P)$. To do this, let $D \in \mathcal{E}(I)$. Then $e_D \in P \cap \text{Lin}(P)^{\perp}$ (by (5.12) and the definition of e_D). Let $x_1, x_1 \in P$ satisfy $e_D = \frac{x_1 + x_2}{2}$. It follows that $\langle a_i, x_1 \rangle = \langle a_i, x_2 \rangle = c_i$ for all $i \in D$, and so $\langle a_i, x_1 - x_2 \rangle = 0$ for all $i \in D$. Since $\{a_i : i \in D\}$ is a maximal linearly independent subset of $\{a_i : i \in I\}$, $\langle a_i, x_1 - x_2 \rangle = 0$ for all $i \in I$. Hence $x_1 - x_2 \in \text{Lin}(P)$. This shows that $e_D \in \text{ext}_E(P)$. The proof is completed.

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