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THE LAGRANGE MULTIPLIER RULE FOR MULTIFUNCTIONS IN BANACH SPACES*

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Abstract. We study general constrained multiobjective optimization problems with objectives being closed multifunctions in Banach spaces. In terms of the coderivatives and normal cones, we provide generalized Lagrange multiplier rules as necessary optimality conditions of the above problems. In an Asplund space setting, sharper results are presented.

Key words. multifunction, normal cone, coderivative, Pareto solution

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1. Introduction.

Let X be a Banach space and $f_i : X \to R \cup \{+\infty\}$ be proper lower semicontinuous functions ($i = 0, 1, \ldots, m$). Many authors (see [2, 3, 4, 16, 29, 30]) studied the following optimization problem with inequality and equality constraints:

$$\begin{aligned} &\min f_0(x), \\ &f_i(x) \le 0, \quad i = 1, \ldots, n, \\ &f_i(x) = 0, \quad i = n+1, \ldots, m, \\ &x \in \Omega. \end{aligned} \tag{1.1}$$

Under some restricted conditions (e.g., each f_i is locally Lipschitz), it is well known, as the Lagrange multiplier rule, that if $\bar{x}$ is a local solution of (1.1), then there exists $\lambda_i \in R$ $(0 \le i \le m)$ such that

$$\begin{aligned} &0 \in \sum_{i=0}^{m} \partial(\lambda_i f_i)(\bar{x}) + N(\Omega, \bar{x}), \\ &\sum_{i=0}^{m} |\lambda_i| = 1 \text{ and } \lambda_i \ge 0, \quad 0 \le i \le n, \end{aligned} \tag{1.2}$$

where $\partial(\lambda_i f_i)$ and $N(\Omega, \bar{x})$ denote the subdifferential and the normal cone (see section 2 for their definitions). Some authors established the so-called fuzzy Lagrange multiplier rule (see [3, 14, 20] and the references therein). The main aim of this paper is to establish the corresponding rules for multifunctions in Banach spaces.

Let X, $Y_0, Y_1, \ldots, Y_m$ be Banach spaces, Ω be a closed subset of X, and $F_i : X \to 2^{Y_i}$ $(i = 0, 1, \ldots, m)$ be closed multifunctions. Let $C_0 \subset Y_0$ be a closed convex cone

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such that $C_0 \neq C_0 \cap -C_0$ (i.e, C_0 is not a linear subspace), which specifies a preorder $\leq_{C_0}$ on Y_0 as follows: for $y_1, y_2 \in Y_0$,

$$y_1 \leq_{C_0} y_2 \text{ if and only if } y_2 - y_1 \in C_0.$$

For $i = 1, \ldots, m$, let C_i be a closed convex cone in Y_i. Consider the following constrained multiobjective optimization problem:

$$\begin{aligned} & C_0 - \min F_0(x), \\ & F_i(x) \cap -C_i \neq \emptyset, \quad i = 1, \ldots, m, \\ & x \in \Omega. \end{aligned} \tag{1.3}$$

Recall that $\bar{a} \in A$ is said to be a Pareto efficient point if $\bar{a} \leq_{C_0} a$ whenever $a \in A$ and $a \leq_{C_0} \bar{a}$, that is,

$$A \cap (\bar{a} - C_0) \subset \bar{a} + C_0 \cap -C_0.$$

We use $\mathrm{E}(A, C_0)$ to denote the set of all Pareto efficient points of A. In the case when C_0 is pointed (i.e., $C_0 \cap -C_0 = \{0\}$),

$$\bar{a} \in E(A, C_0) \Longleftrightarrow A \cap (\bar{a} - C_0) = \{\bar{a}\}.$$

For $\bar{x} \in X$ and $\bar{y} \in F_0(\bar{x})$, we say that $(\bar{x}, \bar{y})$ is a local Pareto solution of the multiobjective optimization problem (1.3) if there exists a neighborhood U of $\bar{x}$ such that

$$\bar{y} \in \mathrm{E}\left(F_0\left[U \cap \Omega \cap \left(\bigcap_{i=1}^{m} F_i^{-1}(-C_i)\right)\right], C_0\right).$$

In the case when each F_i is single-valued, many authors have established sufficient or necessary optimality conditions for Pareto solutions and weak Pareto solutions under some restricted conditions; e.g., the ordering cone has a nonempty interior, the spaces are finite dimensional, and $C_i = R^n_+$ (see [1, 5, 7, 9, 10, 11, 12, 13, 22, 23, 24, 26, 27] and the references therein). In the set-valued setting, in terms of cotangent derivatives Götz and Jahn [8] provided the Lagrange multiplier rule for (1.3) under the convexity assumption. Ye and Zhu [25] and Mordukhovich, Treiman, and Zhu [19] gave some necessary optimality conditions for multiobjective optimization problems with respect to an abstract order in a Euclidean space or Asplund space setting. Recently, the authors [28] studied a unconstrained multiobjective problem with the objective being multifunctions in Banach spaces and, as generalizations of the Fermat rule, presented necessary optimization conditions. In this paper, in a general setting we provide the following fuzzy Lagrange multiplier rule for constrained multiobjective optimization problem (1.3).

Let X, Y_i be Banach spaces, Ω be a closed subset of X, and $F_i : X \to 2^{Y_i}$ be a closed multifunction ($i = 0, 1, \ldots, m$). Suppose that $(\bar{x}, \bar{y}_0)$ is a local Pareto solution of the constrained multiobjective optimization problem (1.3), and let $\bar{y}_i \in F_i(\bar{x}) \cap -C_i$ ($i = 1, \ldots, m$). Then one of the following two assertions holds.

(i) For any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, and $c_i^* \in C_i^+$ such that

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \text{ and } 0 \in \sum_{i=0}^{m} D_c^* F_i(x_i, y_i)(c_i^* + \varepsilon B_{Y_i^*}) + N_c(\Omega, w) + \varepsilon B_{X^*},$$

where B_X denotes the closed unit ball of X, $C_i^+ := \{y^* \in Y_i^* : \langle y^*, c\rangle \geq 0 \ \forall c \in C_i\}$, $N_c(\cdot,\cdot)$ denotes the Clarke normal cone, and $D_c^* F_i(\cdot,\cdot)$ denotes the Mordukhovich coderivative with respect to the Clarke normal cone.

(ii) For any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, $x_i^* \in D_c^* F_i(x_i, y_i)(\varepsilon B_{Y_i^*})$, and $w^* \in N_c(\Omega, w) + \varepsilon B_{X^*}$ such that

$$\|w^*\| + \sum_{i=0}^{m} \|x_i^*\| = 1 \ \text{ and } \ w^* + \sum_{i=0}^{m} x_i^* = 0. \tag{1.4}$$

Using this result, we give some exact Lagrange multiplier rules for (1.3). In the case when X, Y_i are Asplund spaces, these results are sharpened; in particular, we prove the following result (see section 2 for terms undefined).

Let $(\bar{x}, \bar{y}_0)$ be a local Pareto solution of (1.3), and let $\bar{y}_i \in F_i(\bar{x}) \cap -C_i$. Suppose that each F_i is pseudo-Lipschitz around $(\bar{x}, \bar{y}_i)$ and that each C_i is dually compact (e.g., C_i has a nonempty interior). Then there exists $c_i^* \in C_i^+$ such that

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \ \text{ and } \ 0 \in \sum_{i=0}^{m} D^* F_i(\bar{x}, \bar{y}_i)(c_i^*) + N(\Omega, \bar{x}), \tag{1.5}$$

where $D^* F_i(\cdot,\cdot)$ denotes the Mordukhovich coderivative with respect to the limiting normal cone (see section 2 for its definition). Under the condition that X, Y_i are finite dimensional, we provide the following necessity optimality condition of constrained multiobjective optimization problem (1.3).

Let each F_i be a closed multifunction and each C_i be a closed convex cone. Suppose that $(\bar{x}, \bar{y}_0)$ is a local Pareto solution of (1.3). Then, for any $\bar{y}_i \in F_i(\bar{x}) \cap -C_i$, one of the following assertions holds.

(a) There exists $c_i^* \in C_i^+$ such that (1.5) holds.

(b) There exist $x_i^* \in D^* F_i(\bar{x}, \bar{y}_i)(0)$ and $w^* \in N(\Omega, \bar{x})$ such that (1.4) holds.

Let $f_0, f_1, \ldots, f_m$ be as in (1.1). In the special case when $Y_i = R$, $C_i = \{0\}$ for $0 \leq i \leq m$, $F_i(x) = [f_i(x), +\infty)$ for $0 \leq i \leq n$, and $F_i(x) = f_i(x)$ for $n+1 \leq i \leq m$. The above results can be applied to (1.1). In particular, under the assumption that X is an Asplund space and that $f_0, f_1, \cdot, f_n$ are lower semicontinuous and $f_{n+1}, \ldots, f_m$ are continuous, we prove that if $\bar{x}$ is a local solution of (1.1), then one of the following assertions holds.

(i) For any $\varepsilon > 0$ there exist $\lambda_i \in R \setminus \{0\}$, $w \in (\bar{x} + \varepsilon B_X) \cap \Omega$, and $x_i \in \bar{x} + \varepsilon B_X$ with $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ such that $\lambda_i \geq 0$ for $0 \leq i \leq n$, $\sum_{i=0}^{m} |\lambda_i| = 1$, and

$$0 \in \sum_{i=0}^{m} \hat{\partial}(\lambda_i f_i)(x_i) \cap M B_{X^*} + \hat{N}(\Omega, w) \cap M B_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

(ii) For any $\varepsilon > 0$ there exist $w \in (\bar{x} + \varepsilon B_X) \cap \Omega$, $x_i \in \bar{x} + \varepsilon B_X$ with $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$, $\varepsilon_i \in (-\varepsilon, \varepsilon)$, $w^* \in \hat{N}(\Omega, w) + \varepsilon B_{X^*}$, and $x_i^* \in \hat{\partial}(\varepsilon_i f_i)(x_i)$ such that (1.4) holds and $\varepsilon_i > 0$ for $0 \leq i \leq n$.

2. Preliminaries. Throughout this section, we assume that Y is a Banach space. Let $f : Y \to R \cup \{+\infty\}$ be a proper lower semicontinuous function, and let $\text{epi}(f)$ denote the epigraph of f, that is,

$$\text{epi}(f) := \{(y, t) \in Y \times R : \ f(y) \leq t\}.$$

Let $y \in \text{dom}(f)$, let $h \in Y$, and let $f^\circ(y,h)$ denote the generalized directional derivative given by Rockafellar (see [4]), that is,

$$f^\circ(y,h) := \lim_{\varepsilon\downarrow 0} \limsup_{z \xrightarrow{f} y,\, t\downarrow 0} \inf_{w\in h+\varepsilon B_Y} \frac{f(z+tw)-f(z)}{t},$$

where B_Y denotes the closed unit ball of Y, and the expression $z \xrightarrow{f} y$ means $z \to y$ and $f(z) \to f(y)$. It is known that $f^\circ(y,h)$ reduces to Clarke's directional derivative when f is locally Lipschitzian (see [4]). Let

$$\partial_c f(y) := \{y^* \in Y^* :\ \langle y^*, h\rangle \le f^\circ(y,h) \quad \forall h \in Y\}.$$

Let A be a closed subset of Y, and let $N_c(A,a)$ denote Clarke's normal cone of A at a, that is,

$$N_c(A,a) := \begin{cases} \partial_c \delta_A(a), & a \in A, \\ \emptyset, & a \notin A, \end{cases}$$

where δ_A denotes the indicator function of A: $\delta_A(y) = 0$ if $y \in A$ and $\delta_A(y) = +\infty$ otherwise. The following result (see [4, Corollary, p. 52]) presents an important necessity optimality condition in terms of Clarke's subdifferential and normal cone for a nonsmooth constrained optimization problem.

PROPOSITION 2.1. *Let $f : Y \to R$ be a locally Lipschitz function and A be a closed subset of Y. Suppose that f attains its minimum over A at $a \in A$. Then $0 \in \partial_c f(a) + N_c(A,a)$.*

We also need the notion of Fréchet normal cones and that of limiting normal cones. For $\varepsilon \ge 0$, the set of ε-normals to A at a is defined by

$$\hat{N}_\varepsilon(A,a) := \left\{ y^* \in Y^* : \limsup_{y \xrightarrow{A} a} \frac{\langle y^*, y-a\rangle}{\|y-a\|} \le \varepsilon \right\},$$

where $y \xrightarrow{A} a$ means that $y \to a$ with $y \in A$. The set $\hat{N}_0(A,a)$ is simply denoted by $\hat{N}(A,a)$ and is called the Fréchet normal cone to A at a. The limiting Fréchet normal cone to A at a is defined by

$$N(A,a) := \{y^* \in Y^* :\ \exists \varepsilon_n \to 0^+,\ y_n \xrightarrow{A} a,\ y_n^* \xrightarrow{w^*} y^* \text{ with } y_n^* \in \hat{N}_{\varepsilon_n}(A, y_n)\}.$$

In the case when A is convex, it is well known that

$$N_c(A,a) = N(A,a) = \hat{N}(A,a).$$

Recall that the Fréchet subdifferential $\hat{\partial} f(y)$ and the limiting subdifferential $\partial f(y)$ of f at $y \in \text{dom}(f)$ are defined by

$$\hat{\partial} f(y) = \{y^* :\ (y^*, -1) \in \hat{N}(\text{epi}(f), (y, f(y))\}$$

and

$$\partial f(y) := \{y^* \in Y^* :\ (y^*, -1) \in N(\text{epi}(f), (y, f(y)))\},$$

respectively. It is known (see [18]) that

$$\hat{\partial} f(y) := \left\{ y^* \in Y^* : \liminf_{v \to y} \frac{f(v) - f(y) - \langle y^*, v - y \rangle}{\|v - y\|} \geq 0 \right\}.$$

Let $\hat{\partial}^\infty f(y)$ and $\partial^\infty f(y)$ denote, respectively, the singular Fréchet subdifferential and the singular limiting subdifferential of f at y, that is,

$$\hat{\partial}^\infty f(y) = \{y^* : (y^*, 0) \in \hat{N}(\mathrm{epi}(f), (y, f(y))\}$$

and

$$\partial^\infty f(y) := \{y^* \in Y^* : (y^*, 0) \in N(\mathrm{epi}(f), (y, f(y)))\}.$$

Recall that a Banach space Y is called an Asplund space if every continuous convex function defined on an open convex subset D of Y is Fréchet differentiable at each point of a dense G_δ subset of D. It is well known that Y is an Asplund space if and only if every separable subspace of Y has a separable dual. The class of Asplund spaces is well investigated in geometric theory of Banach spaces; see [21] and the references therein. In the case when Y is an Asplund space, Mordukhovich and Shao [18] proved that $\partial f(y) = \limsup_{v \xrightarrow{f} y} \hat{\partial} f(v)$,

$$N(A, a) := \{y^* \in Y^* : \exists y_n \xrightarrow{A} a,\ y_n^* \xrightarrow{w^*} y^* \text{ with } y_n^* \in \hat{N}(A, y_n)\}, \tag{2.1}$$

and $N_c(A, a)$ is the weak* closed convex hull of $N(A, a)$.

In the Asplund space setting, in terms of the Fréchet subdifferential and Fréchet normal cone one has the following necessity optimality condition similar to Proposition 2.1.

PROPOSITION 2.2. *Let Y be an Asplund space and $f : Y \to R$ a locally Lipschitz function, and let A be a closed subset of Y. Suppose that f attains its minimum over A at $a \in A$. Then for any $\varepsilon > 0$ there exist $a_\varepsilon \in a + \varepsilon B_Y$ and $u_\varepsilon \in A \cap (a + \varepsilon B_Y)$ such that*

$$0 \in \hat{\partial} f(a_\varepsilon) + \hat{N}(A, u_\varepsilon) + \varepsilon B_{Y^*}.$$

Proposition 2.2 is due to Fabian [6] (also see [18] for the details).

For $\Phi : X \to 2^Y$, a multifunction from another Banach space X to Y, let $\mathrm{Gr}(\Phi)$ denote the graph of Φ, that is,

$$\mathrm{Gr}(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}.$$

We say that Φ is closed if $\mathrm{Gr}(\Phi)$ is a closed subset of $X \times Y$ and that Φ is convex if $\mathrm{Gr}(\Phi)$ is a convex subset of $X \times Y$. Recall (see [15, 17]) that Φ is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ if there exist a constant $L > 0$, a neighborhood U of $\bar{x}$, and a neighborhood V of $\bar{y}$ such that

$$\Phi(x_1) \cap V \subset \Phi(x_2) + \|x_1 - x_2\| L B_Y \quad \forall x_1, x_2 \in U.$$

For $x \in X$ and $y \in \Phi(x)$, let $\hat{D}^*\Phi(x, y)$, $D^*\Phi(x, y)$ and $D_c^*\Phi(x, y) : Y^* \to 2^{X^*}$ denote the Mordukhovich coderivatives of Φ at (x, y) with respect to the Fréchet, limiting, and Clarke normal cones, respectively, that is,

$$\hat{D}^*\Phi(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in \hat{N}(\mathrm{Gr}(\Phi), (x, y))\} \quad \forall y^* \in Y^*, \tag{2.2}$$

$$D^*\Phi(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N(\mathrm{Gr}(\Phi), (x, y))\} \quad \forall y^* \in Y^*, \tag{2.3}$$

and

$$D_c^*\Phi(x,y)(y^*) := \{x^* \in X^* :\ (x^*, -y^*) \in N_c(\mathrm{Gr}(\Phi),(x,y))\} \quad \forall y^* \in Y^*$$

(see [17, 18]). We will need the following known result.

PROPOSITION 2.3. *Let $\Phi : X \to 2^Y$ be a closed multifunction. Suppose that Φ is pseudo-Lipschitz at $(\bar{x},\bar{y}) \in \mathrm{gr}(\Phi)$. Then there exist constants $L, \delta > 0$ such that*

$$\sup\{\|x^*\| :\ x^* \in \hat{D}^*\Phi(x,y)(y^*)\} \le L\|y^*\|$$

for any $(x,y) \in \mathrm{Gr}(\Phi) \cap (B(\bar{x},\delta) \times B(\bar{y},\delta))$ and any $y^* \in Y^*$.

Proposition 2.3 can be found in Mordukhovich [15]. Moreover, readers can find a simpler proof of Proposition 2.3 in Jourani and Thibault [11].

Let $S_i : M_i \to 2^Y$ $(i = 1,\dots,n)$ be multifunctions from metric spaces M_i with metrics d_i. Recall (see [19]) that $\bar{x}$ is called an extremal point of the system $(S_1,\dots,S_n)$ at $(\bar{s}_1,\dots,\bar{s}_n)$, provided that $\bar{x} \in \bigcap_{i=1}^n S_i(\bar{s}_i)$ and there exists $r > 0$ such that for any $\varepsilon > 0$ there exists $(s_1,\dots,s_n) \in M_1 \times \cdots \times M_n$ with

$$d_i(s_i,\bar{s}_i) \le \varepsilon,\ d(\bar{x}, S_i(s_i)) \le \varepsilon, \quad i = 1,\dots,n, \text{ and } \bigcap_{i=1}^n S_i(s_i) \cap (\bar{x} + rB_Y) = \emptyset.$$

Mordukhovich, Treiman, and Zhu [19] proved the following extended extremal principle.

Theorem MTZ. Let $S_i : M_i \to 2^Y$ be multifunctions from metric spaces (M_i, d_i) to an Asplund space Y, $i = 1,\dots,n$. Assume that $\bar{x}$ is an extremal point of the system $(S_1,\dots,S_n)$ at $(\bar{s}_1,\dots,\bar{s}_n)$, where each S_i is closed-valued around $\bar{s}_i$. Then for any $\sigma > 0$ there exist $s_i \in M_i$, $x_i \in S_i(s_i)$, and $x_i^* \in Y^*$, $i = 1,\dots,n$, such that

$$d_i(s_i,\bar{s}_i) \le \sigma,\ \|x_i - \bar{x}\| \le \sigma,\ x_i^* \in \hat{N}(S_i(s_i), x_i) + \sigma B_{Y^*},\ \sum_{i=1}^n \|x_i^*\| = 1, \text{ and } \sum_{i=1}^n x_i^* = 0.$$

Next we provide a slight improvement of Theorem MTZ, which will be used in the proofs of the main results.

For a natural number n and subsets $A_1,\dots,A_n$ of Y, we define the nonintersection index $\gamma(A_1,\dots,A_n)$ of $A_1,\dots,A_n$ as

$$\gamma(A_1,\dots,A_n) := \inf\left\{\sum_{i=1}^{n-1} \|a_i - a_n\| :\ (a_1,\dots,a_n) \in A_1 \times \cdots \times A_n\right\}.$$

LEMMA 2.1. *Let Y be an Asplund space and $A_1,\dots,A_n$ be closed subsets of Y with $\bigcap_{i=1}^n A_i = \emptyset$. Let $a_i \in A_i$ $(i = 1,\dots,n)$ and $\varepsilon > 0$ such that*

$$\sum_{i=1}^{n-1} \|a_i - a_n\| < \gamma(A_1,\dots,A_n) + \varepsilon.$$

Then for any $\lambda > 0$ there exist $\tilde{a}_i \in A_i$ and $a_i^* \in Y^*$ such that

$$\sum_{i=1}^n \|a_i - \tilde{a}_i\| < \lambda, \quad a_i^* \in \hat{N}(A_i, \tilde{a}_i) + \frac{\varepsilon}{\lambda} B_{Y^*},$$

$$\sum_{i=1}^n \|a_i^*\| = 1 \text{ and } \sum_{i=1}^n a_i^* = 0.$$

Proof. Let the product Y^n be equipped with the norm $|||(x_1,\dots,x_n)||| = \sum_{i=1}^n \|x_i\|$ for any $x_i \in Y$ $(i=1,\dots,n)$, and define $f: Y^n \to R\cup\{+\infty\}$ by

$$f(x_1,\dots,x_n) := \sum_{i=1}^{n-1} \|x_i - x_n\| + \delta_{A_1\times\cdots\times A_n}(x_1,\dots,x_n) \quad \forall (x_1,\dots,x_n)\in Y^n.$$

Then

$$\inf\{f(x_1,\dots,x_n):\ (x_1,\dots,x_n)\in Y^n\} = \gamma(A_1,\dots,A_n),$$

and so, by the assumption,

$$f(a_1,\dots,a_n) < \inf\{f(x_1,\dots,x_n):\ (x_1,\dots,x_n)\in Y^n\} + \varepsilon.$$

Take $\eta \in (0,\ \varepsilon)$ and $\beta\in(0,\ \lambda)$ such that

$$\frac{\eta}{\beta} < \frac{\varepsilon}{\lambda} \ \text{ and } \ f(a_1,\dots,a_n) < \inf\{f(x_1,\dots,x_n):\ (x_1,\dots,x_n)\in Y^n\} + \eta.$$

Then, by the Ekeland variational principle, there exists $\tilde{x}_i \in A_i$ such that

$$\sum_{i=1}^n \|a_i - \tilde{x}_i\| \le \beta \tag{2.4}$$

and

$$f(\tilde{x}_1,\dots,\tilde{x}_n) \le f(x_1,\dots,x_n) + \frac{\eta}{\beta}\sum_{i=1}^n \|x_i - \tilde{x}_i\| \quad \forall (x_1,\dots,x_n)\in Y^n. \tag{2.5}$$

This and the definition of f imply that $(\tilde{x}_1,\dots,\tilde{x}_n)\in A_1\times\cdots\times A_n$. It follows from $\bigcap_{i=1}^n A_i = \emptyset$ that

$$\sum_{i=1}^{n-1} \|\tilde{x}_i - \tilde{x}_n\| > 0. \tag{2.6}$$

We define a continuous convex function ψ by

$$\psi(x_1,\dots,x_n) := \sum_{i=1}^{n-1} \|x_i - x_n\| + \frac{\eta}{\beta}\sum_{i=1}^n \|x_i - \tilde{x}_i\| \quad \forall (x_1,\dots,x_n)\in Y^n.$$

It follows from (2.5) that ψ attains its minimum over $A_1\times\cdots\times A_n$ at $(\tilde{x}_1,\dots,\tilde{x}_n)$. By (2.6) and Proposition 2.2, there exist $\bar{x}_i \in Y$ and $\tilde{a}_i\in A_i$ $(i=1,\dots,n)$ such that

$$\sum_{i=1}^{n-1} \|\bar{x}_i - \bar{x}_n\| > 0, \quad \sum_{i=1}^n \|\tilde{a}_i - \tilde{x}_i\| < \lambda - \beta$$

and

$$0 \in \partial\psi(\bar{x}_1,\dots,\bar{x}_n) + \hat{N}(A_1\times\cdots\times A_n, (\tilde{a}_1,\dots,\tilde{a}_n)) + \left(\frac{\varepsilon}{\lambda} - \frac{\eta}{\beta}\right) B_{Y^*}^n. \tag{2.7}$$

It follows from (2.4) that $\sum_{i=1}^n \|\tilde{a}_i - a_i\| \le \sum_{i=1}^n \|\tilde{a}_i - \tilde{x}_i\| + \sum_{i=1}^n \|\tilde{x}_i - a_i\| < \lambda$. Let

$$\phi(x_1, \dots, x_n) := \sum_{i=1}^{n-1} \|x_i - x_n\| \quad \forall (x_1, \dots, x_n) \in Y^n.$$

Then

$$\partial\psi(\bar{x}_1, \dots, \bar{x}_n) \subset \partial\phi(\bar{x}_1, \dots, \bar{x}_n) + \frac{\eta}{\beta} B_{(Y^n)^*}.$$

This and (2.7) imply that

$$(2.8) \qquad 0 \in \partial\phi(\bar{x}_1, \dots, \bar{x}_n) + \hat{N}(A_1 \times \cdots \times A_n, (\tilde{a}_1, \dots, \tilde{a}_n)) + \frac{\varepsilon}{\lambda} B_{(Y^n)^*}.$$

We claim that

$$(2.9) \quad \partial\phi(\bar{x}_1, \dots, \bar{x}_n) \subset \left\{ (x_1^*, \dots, x_n^*) \in (Y^*)^n : \sum_{i=1}^n x_i^* = 0 \text{ and } \sum_{i=1}^n \|x_i^*\| \ge 1 \right\}.$$

Granting this and noting that

$$\hat{N}(A_1 \times \cdots \times A_n, (\tilde{a}_1, \dots, \tilde{a})) = \hat{N}(A_1, \tilde{a}_1) \times \cdots \times \hat{N}(A_n, \tilde{a}_n)$$

is a cone, it follows from (2.8) that there exists $(a_1^*, \dots, a_n^*) \in (Y^*)^n$ such that

$$a_i^* \in \hat{N}(A_i, \tilde{a}_i) + \frac{\varepsilon}{\lambda} B_{Y^*}, \ \sum_{i=1}^n \|a_i^*\| = 1, \ \text{ and } \ \sum_{i=1}^n a_i^* = 0.$$

It remains to show that (2.9) holds. Let $(x_1^*, \dots, x_n^*) \in \partial\phi(\bar{x}_1, \dots, \bar{x}_n)$. It follows from the convexity of ϕ that for any $h \in Y$,

$$\sum_{i=1}^n \langle x_i^*, h \rangle \le \phi(\bar{x}_1 + h, \dots, \bar{x}_n + h) - \phi(\bar{x}_1, \dots, \bar{x}_n) = 0.$$

This means that $\sum_{i=1}^n x_i^* = 0$. On the other hand,

$$-\sum_{i=1}^{n-1} \langle x_i^*, \bar{x}_i - \bar{x}_n \rangle = \sum_{i=1}^n \langle x_i^*, -\bar{x}_i \rangle \le \phi(0, \dots, 0) - \phi(\bar{x}_1, \dots, \bar{x}_n) = -\sum_{i=1}^{n-1} \|\bar{x}_i - \bar{x}_n\|.$$

Since, as in (2.6), $\sum_{i=1}^{n-1} \|\bar{x}_i - \bar{x}_n\| > 0$, it follows that $\sum_{i=1}^n \|x_i^*\| \ge 1$. This completes the proof. □

Remark. Lemma 2.1 recaptures Theorem MTZ. Indeed, by the assumption of Theorem MTZ, there exists $r > 0$ such that for any $\sigma \in (0, \min\{\frac{r}{2}, r^{\frac{1}{2}}\})$ there exists $(s_1, \dots, s_n) \in M_1 \times \cdots \times M_n$ such that each $S_i(s_i)$ is closed,

$$d_i(s_i, \bar{s}_i) < \sigma, \ d(\bar{x}, S_i(s_i)) < \frac{\sigma^2}{2n}, \quad i = 1, \dots, n, \text{ and } \bigcap_{i=1}^n S_i(s_i) \cap (\bar{x} + rB_Y) = \emptyset.$$

Hence, there exists $u_i \in S_i(s_i)$ such that $\|u_i - \bar{x}\| < \frac{\sigma^2}{2n}$. This implies that

$$\sum_{i=1}^{n-1} \|u_i - u_n\| \le \sum_{i=1}^{n-1} (\|u_i - \bar{x}\| + \|\bar{x} - u_n\|) < \sigma^2,$$

and so

$$\sum_{i=1}^{n-1} \|u_i - u_n\| < \gamma(S_1(s_1) \cap (\bar{x} + rB_Y), \ldots, S_n(s_n) \cap (\bar{x} + rB_Y)) + \sigma^2.$$

Now with $A_i = S_i(s_i) \cap (\bar{x} + rB_Y)$, $a_i = u_i$, $\varepsilon = \sigma^2$, and $\lambda = \sigma$, there exist $\tilde{a}_i \in A_i$ and $a_i^* \in Y^*$ satisfying the properties as stated in Lemma 2.1. Note that $\tilde{a}_i$ lies in the interior of $\bar{x} + rB_Y$, and it follows that $a_i^* \in \hat{N}(S_i(s_i), \tilde{a}_i)$. Thus Theorem MTZ is seen to hold.

Similar to the proof of Lemma 2.1 but applying Proposition 2.1 in place of Proposition 2.2, we have the following result applicable to the case when Y is a general Banach space.

LEMMA 2.2. *Let Y be a Banach space and $A_1, \ldots, A_n$ be closed subsets of Y with $\bigcap_{i=1}^n A_i = \emptyset$. Let $a_i \in A_i$ $(i = 1, \ldots, n)$ and $\varepsilon > 0$ such that*

$$\sum_{i=1}^{n-1} \|a_i - a_n\| \le \gamma(A_1, \ldots, A_n) + \varepsilon.$$

Then for any $\lambda > 0$ there exist $\tilde{a}_i \in A_i$ and $a_i^* \in Y^*$ such that

$$\sum_{i=1}^{n} \|a_i - \tilde{a}_i\| < \lambda, \quad a_i^* \in N_c(A_i, \tilde{a}_i) + \frac{\varepsilon}{\lambda} B_{Y^*},$$

$$\sum_{i=1}^{n} \|a_i^*\| = 1 \text{ and } \sum_{i=1}^{n} a_i^* = 0.$$

3. Fuzzy Lagrange multiplier rules. In this section, we always assume that X, Y_i are Banach spaces (unless stated otherwise), that $C_i \subset Y_i$ is a closed convex cone, and that each multifunction $F_i : X \to 2^{Y_i}$ is closed. Further we assume that the ordering cone C_0 in Y_0 is nontrivial (i.e., C_0 is not a linear subspace). For convenience we define the norm on the product $X \times \prod_{i=0}^m Y_i$ by

$$\|(x, y_0, y_1, \ldots, y_m)\| = \|x\| + \sum_{i=0}^{m} \|y_i\|.$$

In this section we present three fuzzy Lagrange multiplier rules. The first one works on general Banach spaces, while the last two work on Asplund spaces dealing, respectively, with the set-valued and the numeral-valued functions.

THEOREM 3.1. *Let $(\bar{x}, \bar{y}_0)$ be a local Pareto solution of the constrained multiobjective optimization problem* (1.3) *and $\bar{y}_i$ be a point in $F_i(\bar{x}) \cap -C_i$ $(i = 1, \ldots, m)$. Then one of the following assertions holds.*

(i) *For any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, and $c_i^* \in C_i^+$ such that*

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \text{ and } 0 \in \sum_{i=0}^{m} D_c^* F_i(x_i, y_i)(c_i^* + \varepsilon B_{Y_i^*}) \cap MB_{X^*} + N_c(\Omega, w) \cap MB_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

(ii) *For any* $\varepsilon > 0$ *there exist* $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, $x_i^* \in D_c^* F_i(x_i, y_i)(\varepsilon B_{Y_i^*})$, *and* $w^* \in N_c(\Omega, w) + \varepsilon B_{X^*}$ *such that*

$$\|w^*\| + \sum_{i=0}^{m} \|x_i^*\| = 1 \text{ and } w^* + \sum_{i=0}^{m} x_i^* = 0.$$

Proof. By the assumption there exists $\delta > 0$ such that

$$\bar{y}_0 \in E\left(F_0\left[(\bar{x} + \delta B_X) \cap \Omega \cap \left(\bigcap_{i=1}^{m} F_i^{-1}(-C_i)\right)\right], C_0\right). \tag{3.1}$$

Since the ordering cone C_0 is not a subspace of Y_0, there exists $c_0 \in C_0$ with $\|c_0\| = 1$ such that

$$c_0 \notin -C_0. \tag{3.2}$$

For any natural number k, let $s_k := \frac{1}{(m+2)k^2}$, and consider the following sets in the product space $X \times \prod_{j=0}^{m} Y_j$:

$$A_i := \left\{(x, y_0, y_1, \ldots, y_m) \in X \times \prod_{j=0}^{m} Y_j : (x, y_i) \in \mathrm{Gr}(F_i)\right\}, \quad i = 0, 1, \ldots, m,$$

and

$$A_{m+1} := ((\bar{x} + \delta B_X) \cap \Omega) \times (\bar{y}_0 - s_k c_0 - C_0) \times \prod_{i=1}^{m} (\bar{y}_i - C_i).$$

Then $\bigcap_{i=0}^{m+1} A_i = \emptyset$. Indeed, if this is not the case, then there exist $x' \in X$ and $y_i' \in F_i(x')$ $(i = 0, 1, \ldots, m)$ such that

$$x' \in (\bar{x} + \delta B_X) \cap \Omega,\ y_0' \le_{C_0} \bar{y}_0 - s_k c_0, \text{ and } y_i' \in \bar{y}_i - C_i (\subset -C_i), \quad i = 1, \ldots, m.$$

Hence, $x' \in (\bar{x} + \delta B_X) \cap \Omega \cap \left(\bigcap_{i=1}^{m} F_i^{-1}(-C_i)\right)$, and so

$$y_0' \in F_0\left[(\bar{x} + \delta B_X) \cap \Omega \cap \left(\bigcap_{i=1}^{m} F_i^{-1}(-C_i)\right)\right].$$

It follows from (3.1) that $\bar{y}_0 \le_{C_0} y_0'$, and so $\bar{y}_0 \le_{C_0} \bar{y}_0 - s_k c_0$. This implies that $c_0 \in -C_0$, contradicting (3.2). Let

$$a_0 = a_1 = \cdots = a_m = (\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_m) \text{ and } a_{m+1} = (\bar{x}, \bar{y}_0 - s_k c_0, \bar{y}_1, \ldots, \bar{y}_m).$$

Then

$$\sum_{i=0}^{m} \|a_i - a_{m+1}\| = (m+1) s_k < \frac{1}{k^2} \le \gamma(A_0, A_1, \ldots, A_{m+1}) + \frac{1}{k^2}.$$

By Lemma 2.2 (applied to the family $\{A_0, A_1, \ldots, A_{m+1}\}$ and the constants $\varepsilon = \frac{1}{k^2}$, $\lambda = \frac{1}{k}$), there exist

$$\tilde{a}_i(k) := (x_i(k), y_{i,0}(k), y_{i,1}(k), \ldots, y_{i,m}(k)) \in X \times \prod_{j=0}^{m} Y_j$$

and

$$(x_i^*(k), y_{i,0}^*(k), y_{i,1}^*(k), \dots, y_{i,m}^*(k)) \in X^* \times \prod_{j=0}^{m} Y_j^*$$

$(i = 0, 1, \dots, m+1)$ such that

$$\sum_{i=0}^{m+1} \|\tilde{a}_i(k) - a_i\| = \sum_{i=0}^{m} \left(\|x_i(k) - \bar{x}\| + \sum_{j=0}^{m} \|y_{i,j}(k) - \bar{y}_j\| \right) \tag{3.3}$$

$$+ \ \|x_{m+1}(k) - \bar{x}\| + \|y_{m+1,0}(k) - (\bar{y}_0 - s_k c_0)\| + \sum_{j=1}^{m} \|y_{m+1,j}(k) - \bar{y}_j\| < \frac{1}{k},$$

$$(x_i^*(k), y_{i,0}^*(k), \dots, y_{i,m}^*(k)) \in N_c(A_i, \tilde{a}_i(k)) + \frac{1}{k}\left(B_{X^*} \times \prod_{j=0}^{m} B_{Y_j^*} \right), \tag{3.4}$$

$$\sum_{i=0}^{m+1} \max\{\|x_i^*(k)\|, \ \max\{\|y_{i,j}^*(k)\| : \ j = 0, 1, \dots, m\}\} = 1, \tag{3.5}$$

and

$$\sum_{i=0}^{m+1} (x_i^*(k), y_{i,0}^*(k), y_{i,1}^*(k), \dots, y_{i,m}^*(k)) = 0. \tag{3.6}$$

By the definitions of A_{m+1} and $\tilde{a}_{m+1}(k)$, we see that $N_c(A_{m+1}, \tilde{a}_{m+1}(k))$ is equal to the following product:

$$N_c((\bar{x}+\delta B_X)\cap\Omega, x_{m+1}(k)) \times N_c(\bar{y}_0 - s_k c_0 - C_0, y_{m+1,0}(k)) \times \prod_{j=1}^{m} N_c(\bar{y}_j - C_j, y_{m+1,j}(k)).$$

By well-known relations

$$N_c(\bar{y}_0 - s_k c_0 - C_0, y_{m+1,0}(k)) \subset C_0^+ \ \text{ and } \ N_c(\bar{y}_j - C_j, y_{m+1,j}(k)) \subset C_j^+ \ (1 \le j \le m),$$

it follows that

$$N_c(A_{m+1}, \tilde{a}_{m+1}(k)) \subset N_c((\bar{x} + \delta B_X) \cap \Omega, x_{m+1}(k)) \times \prod_{j=0}^{m} C_j^+.$$

We do the above for every natural number k, and by (3.3) we assume without loss of generality that $\bar{x} + \delta B_X$ is a neighborhood of $x_{m+1}(k)$, and so $N_c((\bar{x} + \delta B_X) \cap \Omega, x_{m+1}(k)) = N_c(\Omega, x_{m+1}(k))$. Hence,

$$N_c(A_{m+1}, \tilde{a}_{m+1}(k)) \subset N_c(\Omega, x_{m+1}(k)) \times \prod_{j=0}^{m} C_j^+.$$

This and (3.4) imply that there exists $(c_0^*(k), c_1^*(k), \dots, c_m^*(k)) \in \prod_{j=0}^{m} C_j^+$ such that

$$x_{m+1}^*(k) \in N_c(\Omega, x_{m+1}(k)) + \frac{1}{k} B_{X^*} \tag{3.7}$$

and

$$\|y^*_{m+1,j}(k) - c^*_j(k)\| \le \frac{1}{k}, \quad j = 0, 1, \ldots, m. \tag{3.8}$$

Moreover, for $0 \le i \le m$, we have by the definition of A_i and $\tilde{a}_i(k)$ that

(3.9)

$$N_c(A_i, \tilde{a}_i(k)) \\ = \{(x^*, y^*_0, \ldots, y^*_m) : (x^*, y^*_i) \in N_c(\mathrm{Gr}(F_i), (x_i(k), y_{i,i}(k))) \text{ and } y^*_j = 0 \quad \forall j \ne i\}.$$

This and (3.4) imply that for $0 \le i \le m$,

$$x^*_i(k) \in D^*_c F_i(x_i(k), y_{i,i}(k)) \left(-y^*_{i,i}(k) + \frac{1}{k} B_{Y^*_i}\right) + \frac{1}{k} B_{X^*} \tag{3.10}$$

and

$$\|y^*_{i,j}(k)\| \le \frac{1}{k}, \quad 0 \le j \le m \text{ and } j \ne i. \tag{3.11}$$

By (3.6), (3.8), and (3.11), one has

$$-y^*_{i,i}(k) = y^*_{m+1,i}(k) + \sum_{l=0, l\ne i}^{m} y^*_{l,i}(k) \in c^*_i(k) + \frac{m+1}{k} B_{Y^*_i}, \quad i = 0, 1, \ldots, m. \tag{3.12}$$

This and (3.10) imply that for $i = 0, 1, \ldots, m$,

$$x^*_i(k) \in D^*_c F_i(x_i(k), y_{i,i}(k)) \left(c^*_i(k) + \frac{m+2}{k} B_{Y^*_i}\right) + \frac{1}{k} B_{X^*}. \tag{3.13}$$

In the case when $\{\sum_{j=0}^m \|c^*_j(k)\|\}$ does not converge to 0, without loss of generality we assume that there exists $r > 0$ such that $\sum_{j=0}^m \|c^*_j(k)\| > r$ for all k (passing to subsequences if necessary). It follows from (3.13), (3.7), and (3.6) that

$$\frac{x^*_i(k)}{\sum\limits_{j=0}^m \|c^*_j(k)\|} \in D^*_c F_i(x_i(k), y_{i,i}(k)) \left(\frac{c^*_i(k)}{\sum\limits_{j=0}^m \|c^*_j(k)\|} + \frac{m+2}{rk} B_{Y^*_i}\right) + \frac{1}{rk} B_{X^*}, \quad 0 \le i \le m,$$

$$\frac{x^*_{m+1}(k)}{\sum\limits_{j=0}^m \|c^*_j(k)\|} \in N_c(\Omega, x_{m+1}(k)) + \frac{1}{rk} B_{X^*} \quad \text{and} \quad \sum_{i=0}^{m+1} \frac{x^*_i(k)}{\sum\limits_{j=0}^m \|c^*_j(k)\|} = 0.$$

By virtue of (3.3) and (3.5) and by considering large enough k, it follows that (i) holds with $M = \frac{m+2}{r}$.

Next we consider the case when $t_k := \sum_{j=0}^m \|c^*_j(k)\| \to 0$. In this case, (3.8) implies that

$$y^*_{m+1,j}(k) \to 0 \text{ for } j = 0, 1, \ldots, m.$$

It follows from (3.11), (3.12), and (3.5) that $\sum_{i=0}^{m+1} \|x^*_i(k)\| \to 1$. Thus, by (3.13), (3.7), and (3.6), there exist

$$\tilde{x}^*_i(k) \in D^*_c F_i(x_i(k), y_{i,i}(k)) \left(c^*_i(k) + \frac{m+2}{k} B_{Y^*_i}\right) \text{ for } i = 0, 1, \ldots, m$$

and

$$\tilde{x}^*_{m+1}(k) \in N_c(\Omega, x_{m+1}(k)) + \frac{m+2}{k} B_{X^*}$$

such that

$$r_k := \sum_{i=0}^{m+1} \|\tilde{x}^*_i(k)\| \to 1 \text{ and } \sum_{i=0}^{m+1} \tilde{x}^*_i(k) = 0.$$

Therefore, for all k large enough,

$$\frac{\tilde{x}^*_i(k)}{r_k} \in D^*_c F_i(x_i(k), y_{i,i}(k)) \left(\left(\frac{c^*_i(k)}{r_k} + \frac{m+2}{kr_k} \right) B_{Y^*_i} \right),$$

$$\frac{\tilde{x}^*_{m+1}(k)}{r_k} \in N_c(\Omega, x_{m+1}(k)) + \frac{m+2}{kr_k} B_{X^*},$$

$$\sum_{i=0}^{m+1} \left\| \frac{\tilde{x}^*_i(k)}{r_k} \right\| = 1 \text{ and } \sum_{i=0}^{m+1} \frac{\tilde{x}^*_i(k)}{r_k} = 0.$$

Noting that $r_k \to 1$ and $\|c^*_i(k)\| \leq t_k \to 0$, this implies that (ii) holds, and the proof is completed. ∎

In the special case when $F_i(x) = 0$ for all $x \in X$ and $i = 1, \ldots, m$, (1.3) reduces to the following problem:

$$C_0 - \min F_0(x), \quad x \in \Omega, \tag{3.14}$$

and $D^*_c F_i(x, 0)(y^*_i) = 0$ for all $(x, y^*_i) \in X \times Y^*_i$ and $i = 1, \ldots, m$. Thus, the following corollary is an immediate consequence of Theorem 3.1 and recaptures [28, Theorem 3.1] by putting our $\Omega = X$.

COROLLARY 3.1. *Let $(\bar{x}, \bar{y})$ be a local Pareto solution of the constrained multiobjective optimization problem* (3.14). *Then one of the following two assertions holds.*

(i) *For any $\varepsilon > 0$ there exist $u \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y \in F_0(u) \cap (\bar{y} + \varepsilon B_Y)$, and $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$0 \in D^*_c F_0(u, y)(c^* + \varepsilon B_{Y^*}) \cap M B_{X^*} + N_c(\Omega, w) \cap M B_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

(ii) *For any $\varepsilon > 0$ there exist $u \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y \in F_0(u) \cap (\bar{y} + \varepsilon B_Y)$, and $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in D^*_c F_0(u, y)(\varepsilon B_{Y^*}) \cap (-N_c(\Omega, w) + \varepsilon B_{X^*}).$$

When X and each Y_i are Asplund spaces, Theorem 3.1 can be strengthened to the following theorem, Theorem 3.2, in which D^*_c and $N_c(\Omega, \cdot)$ are replaced, respectively, by the Fréchet coderivative $\hat{D}^*$ and the Fréchet normal cone $\hat{N}(\Omega, \cdot)$ (recall that $\hat{N}(A, a) \subset N(A, a)$ and $N_c(A, a)$ is the weak*-closed convex hull of $N(A, a)$). The proof is the same as the proof of Theorem 3.1, but use Lemma 2.1 in place of Lemma 2.2.

THEOREM 3.2. *Let $(\bar{x}, \bar{y}_0)$ be a local Pareto solution of the constrained multiobjective optimization problem* (1.3) *and $\bar{y}_i$ be a point in $F_i(\bar{x}) \cap -C_i$ $(i = 1, \ldots, m)$.*

Suppose that X and Y_i $(i = 0, 1, \ldots, m)$ are Asplund spaces. Then one of the following assertions holds.

(i) *For any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, and $c_i^* \in C_i^+$ such that*

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \text{ and } 0 \in \sum_{i=0}^{m} \hat{D}^* F_i(x_i, y_i)(c_i^* + \varepsilon B_{Y_i^*}) \cap M B_{X^*} + \hat{N}(\Omega, w) \cap M B_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

(ii) *For any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, $x_i^* \in \hat{D}^* F_i(x_i, y_i)(\varepsilon B_{Y_i^*})$, and $w^* \in \hat{N}(\Omega, w) + \varepsilon B_{X^*}$ such that*

$$\|w^*\| + \sum_{i=0}^{m} \|x_i^*\| = 1 \text{ and } w^* + \sum_{i=0}^{m} x_i^* = 0.$$

Next we prove that (ii) in Theorem 3.2 cannot happen when each F_i is pseudo-Lipschitz at $(\bar{x}, \bar{y}_i)$.

COROLLARY 3.2. *Let $(\bar{x}, \bar{y}_0)$ be a local Pareto solution of the constrained multi-objective optimization problem* (1.3) *and $\bar{y}_i$ be a point in $F_i(\bar{x}) \cap -C_i$ $(i = 1, \ldots, m)$. Suppose that X and Y_i $(i = 0, 1, \ldots, m)$ are Asplund spaces and that each F_i is pseudo-Lipschitz at $(\bar{x}, \bar{y}_i)$. Then for any $\varepsilon > 0$ there exist $x_i \in \bar{x} + \varepsilon B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon B_X)$, $y_i \in F_i(x_i) \cap (\bar{y}_i + \varepsilon B_{Y_i})$, and $c_i^* \in C_i^+$ such that*

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \text{ and } 0 \in \sum_{i=0}^{m} \hat{D}^* F_i(x_i, y_i)(c_i^* + \varepsilon B_{Y_i^*}) \cap M B_{X^*} + \hat{N}(\Omega, w) \cap M B_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

Proof. Since each F_i is pseudo-Lipschitz at $(\bar{x}, \bar{y}_i)$, Proposition 2.3 implies that there exist constants $L, \delta > 0$ such that for any $(x, y_i) \in \mathrm{Gr}(F_i) \cap (B(\bar{x}, \delta) \times B(\bar{y}_i, \delta))$ and $y_i^* \in Y^*$,

$$\sup\{\|x^*\| : x^* \in \hat{D}^* F_i(x, y_i)(y_i^*)\} \leq L \|y_i^*\|. \tag{3.15}$$

We need only show that (i) of Theorem 3.2 holds. If this is not the case, Theorem 3.2 implies that there exist

$$x_i \in B(\bar{x}, \delta), \ w \in \Omega \cap B(\bar{x}, \delta), \ y_i \in F_i(x_i) \cap B(\bar{y}_i, \delta), \tag{3.16}$$

$$x_i^* \in \hat{D}^* F_i(x_i, y_i) \left(\frac{B_{Y_i^*}}{4(m+1)L} \right) \text{ and } w^* \in \hat{N}(\Omega, w) + B_{X^*} \tag{3.17}$$

such that

$$\|w^*\| + \sum_{i=0}^{m} \|x_i^*\| = 1 \text{ and } w^* + \sum_{i=0}^{m} x_i^* = 0. \tag{3.18}$$

By (3.15), (3.16), and (3.17), one has

$$\left\| \sum_{i=0}^{m} x_i^* \right\| \leq \sum_{i=0}^{m} \|x_i^*\| \leq \frac{1}{4},$$

contradicting (3.18). This completes the proof. ∎

Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function and $F(x) = [f(x),\ +\infty)$ for all $x \in X$. Then F is closed and $\mathrm{Gr}(F) = \mathrm{epi}(f)$. Recall (see [14, Lemma 2.2]) that if $r \in F(\bar{x})$ and X is an Asplund space, then the following assertions hold.

(α) $\lambda \neq 0$ and $x^* \in \hat{D}^*F(\bar{x}, r)(\lambda) \Longleftrightarrow \lambda > 0,\ r = f(\bar{x})$, and $x^* \in \hat{\partial}(\lambda f)(\bar{x})$.

(β) For any $x^* \in \hat{D}^*F(\bar{x}, r)(0)$ there exist sequences $\{x_k\}$, $\{x_k^*\}$, and $\{\lambda_k\}$ such that

$$x_k^* \in \hat{\partial}(\lambda_k f)(x_k),\ (x_k, f(x_k)) \to (\bar{x}, f(\bar{x})),\ \lambda_k \downarrow 0, \text{ and } \|x_k^* - x^*\| \to 0.$$

Let $g : X \to R$ be a continuous function and $G(x) = \{g(x)\}$ for all $x \in X$. The following assertions are known (see [14, Lemma 2.3]).

(α') $\hat{D}^*G(x, g(x))(\lambda) = \partial(\lambda g)(x)$ for any $\lambda \neq 0$.

(β') $x^* \in \hat{D}^*G(x, g(x))(0)$ if and only if there exist sequences $\{x_k\}$, $\{x_k^*\}$, and $\{t_k\}$ such that

$$x_k^* \in \hat{\partial}(t_k g)(x_k) \cup \hat{\partial}(-t_k g)(x_k),\ (x_k, g(x_k)) \to (\bar{x}, g(\bar{x})),\ t_k \downarrow 0, \text{ and } \|x_k^* - x^*\| \to 0.$$

As an application of Theorem 3.2, now we can establish fuzzy necessary optimality conditions for scalar-objective optimization problem (1.1).

THEOREM 3.3. *Let X be an Asplund space and Ω be a closed subset of X. Let $f_0, f_1, \ldots, f_n : X \to R \cup \{+\infty\}$ be proper lower semicontinuous and $f_{n+1}, \ldots, f_m : X \to R$ be continuous. Suppose that $\bar{x}$ is a local solution of* (1.1). *Then one of the following assertions hold.*

(i) *For any $\varepsilon > 0$ there exist $\lambda_i \in R \setminus \{0\}$, $w \in (\bar{x} + \varepsilon B_X) \cap \Omega$, and $x_i \in \bar{x} + \varepsilon B_X$ with $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ such that $\lambda_i > 0$ for $0 \leq i \leq n$, $\sum_{i=0}^m |\lambda_i| = 1$, and*

$$0 \in \sum_{i=0}^m \hat{\partial}(\lambda_i f_i)(x_i) \cap MB_{X^*} + \hat{N}(\Omega, w) \cap MB_{X^*} + \varepsilon B_{X^*},$$

where $M > 0$ is a constant independent of ε.

(ii) *For any $\varepsilon > 0$ there exist $w \in (\bar{x} + \varepsilon B_X) \cap \Omega$, $x_i \in \bar{x} + \varepsilon B_X$ with $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$, $\varepsilon_i \in (-\varepsilon,\ \varepsilon) \setminus \{0\}$, $w^* \in \hat{N}(\Omega, w) + \varepsilon B_{X^*}$, and $x_i^* \in \hat{\partial}(\varepsilon_i f_i)(x_i)$ such that $\varepsilon_i > 0$ for $0 \leq i \leq n$,*

$$\|w^*\| + \sum_{i=0}^m \|x_i^*\| = 1 \text{ and } w^* + \sum_{i=0}^m x_i^* = 0.$$

Proof. Let ε be an arbitrary positive number. By the lower semicontinuity assumption, there exists $\delta \in (0,\ \frac{1}{2})$ such that

$$f_i(\bar{x}) - \varepsilon < f_i(x) \text{ for any } x \in \bar{x} + \delta B_X \text{ and } i = 0, 1, \ldots, n. \tag{3.19}$$

Let $Y_0 = Y_1 = \cdots = Y_m = R$. Let $C_i = R^+$, $F_i(x) = [f_i(x),\ +\infty)$ for $i = 0, 1, \ldots, n$ and $C_i = \{0\}$, $F_i(x) = \{f_i(x)\}$ for $i = n+1, \ldots, m$. Then, each F_i is closed, $(\bar{x}, \bar{y}_0)$ is a local Pareto solution of (1.3), and $\bar{y}_i := f_i(\bar{x}) \in F_i(\bar{x}) \cap -C_i$ for $i = 1, \ldots, m$. Hence, one of the assertions (i) and (ii) in Theorem 3.2 holds. It suffices to show that (i) in Theorem 3.2$\Longrightarrow$(i) and (ii) in Theorem 3.2$\Longrightarrow$(ii). As the arguments are similar, we shall prove only that the implication (i) in Theorem 3.2$\Longrightarrow$(i). Suppose that (i) in Theorem 3.2 holds. Let $\sigma \in (0,\ \min\{\frac{\varepsilon}{4}, \delta\})$, and take ($\alpha$) into account. Then there exist $w \in (\bar{x} + \sigma B_X) \cap \Omega$, $(u_i, r_i) \in (\bar{x} + \sigma B_X) \times (f_i(\bar{x}) - \sigma,\ f_i(\bar{x}) + \sigma)$, and $s_i \in R$ such that

$$r_i \geq f_i(u_i) \text{ for } 0 \leq i \leq n, \quad r_i = f_i(u_i) \text{ for } n+1 \leq i \leq m,$$

$$s_i \geq 0 \text{ for } i = 0, 1, \ldots, n, \quad \sum_{i=0}^{m} |s_i| \geq 1 - \sigma, \tag{3.20}$$

and

$$0 \in \sum_{i=0}^{m} \hat{D}^* F_i(u_i, r_i)(s_i) \cap K B_{X^*} + \hat{N}(\Omega, w) \cap K B_{X^*} + \sigma B_{X^*}, \tag{3.21}$$

where $K > 0$ is a constant. By (3.19), one has

$$f_i(\bar{x}) - \varepsilon < f(u_i) \leq r_i < f_i(\bar{x}) + \sigma < f_i(\bar{x}) + \varepsilon \text{ for } i = 0, 1, \ldots, n. \tag{3.22}$$

Take $u_i^* \in \hat{D}F_i(u_i, r_i)(s_i) \cap K B_{X^*}$ (by (3.21)) such that

$$-\sum_{i=0}^{m} u_i^* \in \hat{N}(\Omega, w) \cap K B_{X^*} + \sigma B_{X^*}. \tag{3.23}$$

Let $I_0 := \{0 \leq i \leq m :\ s_i = 0\}$. It follows from (α) and (α') that

$$u_i^* \subset \hat{\partial}(s_i f_i)(u_i) \cap K B_{X^*} \text{ for any } i \in \{0, 1, \ldots, m\} \setminus I_0. \tag{3.24}$$

For any $i \in \{0, 1, \ldots, n\} \cap I_0$, (3.22) and (β) imply that there exist $\tilde{u}_i \in u_i + \sigma B_X$ with $|f_i(\tilde{u}_i) - f_i(u_i)| < \varepsilon - |f_i(u_i) - f_i(\bar{x})|$, $t_i > 0$, and $x_i^* \in \hat{\partial}(t_i f_i)(\tilde{u}_i)$ such that $\|x_i^* - u_i^*\| < \frac{\sigma}{m}$. Hence, for any $i \in \{0, 1, \ldots, n\} \cap I_0$,

$$\|\tilde{u}_i - \bar{x}\| \leq \|\tilde{u}_i - u_i\| + \|u_i - \bar{x}\| \leq 2\sigma < \varepsilon, \quad |f_i(\tilde{u}_i) - f_i(\bar{x})| < \varepsilon \tag{3.25}$$

and

$$u_i^* \subset \hat{\partial}(t_i f_i)(\tilde{u}_i) \cap \left(K + \frac{1}{m}\right) B_{X^*} + \frac{\sigma}{m} B_{X^*}. \tag{3.26}$$

Moreover, for any $j \in \{n+1, \ldots, m\} \cap I_0$, (β') implies that there exist $\tilde{u}_j \in u_j + \sigma B_X$ with $|f_j(\tilde{u}_j) - f_j(u_j)| < \sigma$, $t_j \in R \setminus \{0\}$, and $x_j^* \in \hat{\partial}(t_j f_j)(\tilde{u}_j)$ such that $\|x_j^* - u_j^*\| < \frac{\sigma}{m}$. Hence, for any $j \in \{n+1, \ldots, m\} \cap I_0$,

$$\|\tilde{u}_j - \bar{x}\| < 2\sigma < \varepsilon, \quad |f_j(\tilde{u}_j) - f_j(\bar{x})| < 2\sigma < \varepsilon \tag{3.27}$$

and

$$u_j^* \subset \hat{\partial}(t_j f_j)(\tilde{u}_j) \cap \left(K + \frac{1}{m}\right) B_{X^*} + \frac{\sigma}{m} B_{X^*}. \tag{3.28}$$

Let $\eta := \sum_{i=0}^{m} |s_i| + \sum_{i \in I_0} |t_i|$, $\lambda_i := \frac{s_i}{\eta}$ if $i \in \{0, 1, \ldots, m\} \setminus I_0$, and $\lambda_i := \frac{t_i}{\eta}$ if $i \in I_0$, and let $x_i := u_i$ if $i \in \{0, 1, \ldots, m\} \setminus I_0$ and $x_i := \tilde{u}_i$ if $i \in I_0$. Then

$$\eta \geq 1 - \sigma > \frac{1}{2},\ \lambda_i > 0 \text{ for } 0 \leq i \leq n,\ \sum_{i=0}^{m} |\lambda_i| = 1,$$

and dividing (3.23), (3.24), (3.26), and (3.28) by η, it follows that

$$0 \in \sum_{i=0}^{m} \hat{\partial}(\lambda_i f_i)(u_i) \cap \left(2K + \frac{2}{m}\right) B_{X^*} + \hat{N}(\Omega, w) \cap 2K B_{X^*} + \varepsilon B_{X^*}.$$

It follows from (3.25) and (3.27) that (i) holds with $M = 2K + \frac{2}{m}$. The proof is completed. □

4. Lagrange multiplier rules. In this section, we provide some exact Lagrange multiplier rules for the constrained multiobjective optimization problem (1.3). We will need the following notions. Recall (see [28]) that a closed convex cone C in X is dually compact if there exists a compact subset K of X such that

$$C^+ \subset \{x^* \in X^* :\ \|x^*\| \le \max\{\langle x^*, x\rangle :\ x \in K\}\}. \tag{4.1}$$

This condition is trivially satisfied if X is finite dimensional (because one can then take $K = B_X$). Note that if C has a nonempty interior, then there exists $c_0 \in C$ such that

$$C^+ \subset \{x^* \in X^* :\ \|x^*\| \le \langle x^*, c_0\rangle\}.$$

Thus,

$$\mathrm{int}(C) \ne \emptyset \Longrightarrow C \text{ is dually compact.}$$

It is known that if C is dually compact, then

$$c_n^* \in C^+ \ \text{ and } \ c_n^* \xrightarrow{w^*} 0 \Longrightarrow c_n^* \to 0. \tag{4.2}$$

The concept C being dually compact is closely related to the locally compact concept introduced in Loewen [12] (see [28, Proposition 3.1] for the details).

Following Mordukhovich [15] and Mordukhovich and Shao [17], we say that a multifunction Φ from X to another Banach space Y is partially sequentially normally compact at $(x, y) \in \mathrm{Gr}(\Phi)$ if for any (generalized) sequence $\{(x_n, y_n, x_n^*, y_n^*)\}$ satisfying

$$x_n^* \in \hat{D}^*\Phi(x_n, y_n)(y_n^*),\ \ (x_n, y_n) \to (x, y),\ \|y_n^*\| \to 0, \text{ and } x_n^* \xrightarrow{w^*} 0$$

one has $\|x_n^*\| \to 0$.

Clearly, Φ is automatically partially sequentially normally compact at each point of $\mathrm{Gr}(\Phi)$ if X is finite dimensional. Moreover, Proposition 2.3 implies that Φ is partially sequentially normally compact at $(x, y) \in \mathrm{Gr}(\Phi)$ if Φ is pseudo-Lipschitz at (x, y).

In the remainder of this paper, we make the following blanket assumptions.

Assumption 4.1. Each F_i is a closed multifunction.

Assumption 4.2. $(\bar{x}, \bar{y}_0) \in \mathrm{Gr}(F_0)$ is a local Pareto solution of the constrained multiobjective optimization problem (1.3) and $\bar{y}_i \in F_i(\bar{x}) \cap -C_i$ $(1 \le i \le m)$.

We first consider the case when X, Y_i are Asplund spaces (thus, in particular (2.1) is valid in these spaces).

THEOREM 4.1. *Let Assumptions* 4.1 *and* 4.2 *hold and* X, Y_i *be Asplund spaces. Suppose that each* C_i *is dually compact and that each* F_i *is partially sequentially normally compact at* $(\bar{x}, \bar{y}_i)$. *Then one of the following assertions holds.*

(i) *There exists* $c_i^* \in C_i^+$ *such that*

$$\sum_{i=0}^m \|c_i^*\| = 1 \ \text{ and } \ 0 \in \sum_{i=0}^m D^*F_i(\bar{x}, \bar{y}_i)(c_i^*) + N(\Omega, \bar{x}).$$

(ii) *There exist* $x_i^* \in D^*F_i(\bar{x}, \bar{y}_i)(0)$ *and* $w^* \in N(\Omega, \bar{x})$ *such that*

$$\|w^*\| + \sum_{i=0}^m \|x_i^*\| = 1 \ \text{ and } \ w^* + \sum_{i=0}^m x_i^* = 0.$$

Proof. Since X, Y_i are Asplund spaces, Assumptions 4.1 and 4.2 imply that one of the assertions (i) and (ii) in Theorem 3.2 holds. Suppose that the assertion (i) in Theorem 3.2 holds. Then, for any natural number k there exist

$$(x_i(k), y_i(k)) \in \mathrm{Gr}(F_i) \cap \left(\left(\bar{x} + \frac{1}{k} B_X \right) \times \left(\bar{y}_i + \frac{1}{k} B_{Y_i} \right) \right), \tag{4.3}$$

$$w(k) \in \left(\bar{x} + \frac{1}{k} B_X \right) \cap \Omega \ \text{ and } \ c_i^*(k) \in C_i^+ \tag{4.4}$$

such that

$$\sum_{i=0}^{m} \|c_i^*(k)\| = 1 \tag{4.5}$$

and

$$\begin{aligned} 0 \in \sum_{i=0}^{m} \hat{D}^* F_i(x_i(k), y_i(k)) \left(c_i^*(k) + \frac{1}{k} B_{Y_i^*} \right) \cap M B_{X^*} \\ + \hat{N}(\Omega, w(k)) \cap M B_{X^*} + \frac{1}{k} B_{X^*}, \end{aligned} \tag{4.6}$$

where $M > 0$ is a constant independent of k. Hence there exist bounded sequences $\{x_i^*(k)\}$ and $\{x^*(k)\}$ such that

$$x_i^*(k) \in \hat{D}^* F_i(x_i(k), y_i(k)) \left(c_i^*(k) + \frac{1}{k} B_{Y_i^*} \right),$$

$$x^*(k) \in \hat{N}(\Omega, w(k)) \quad \text{and} \quad x^*(k) + \sum_{i=0}^{m} x_i^*(k) \to 0.$$

Since a bounded set in a dual space is relatively weak* compact, without loss of generality we can assume that

$$x_i^*(k) \xrightarrow{w^*} x_i^* \ \text{ and } \ c_i^*(k) \xrightarrow{w^*} c_i^* \ (i = 0, 1, \ldots, m).$$

It follows from (2.1), (4.3), and (4.4) that

$$0 \in \sum_{i=0}^{m} D^* F_i(\bar{x}, \bar{y}_i)(c_i^*) + N(\Omega, \bar{x}).$$

Noting that $\sum_{i=0}^{m} \|c_i^*\| \neq 0$ by (4.2) and (4.5), this implies that (i) is true.

Next suppose that assertion (ii) in Theorem 3.2 holds. Then for any natural number k there exist

$$(x_i(k), y_i(k)) \in \mathrm{Gr}(F_i) \cap \left(\left(\bar{x} + \frac{1}{k} B_X \right) \times \left(\bar{y}_i + \frac{1}{k} B_{Y_i} \right) \right), \ w(k) \in \left(\bar{x} + \frac{1}{k} B_X \right) \cap \Omega, \tag{4.7}$$

$$x_i^*(k) \in \hat{D}^* F_i(x_i(k), y_i(k)) \left(\frac{1}{k} B_{Y_i^*} \right) \ \text{ and } \ x^*(k) \in \hat{N}(\Omega, w(k)) \tag{4.8}$$

such that

$$\|x^*(k)\| + \sum_{i=0}^{m} \|x_i^*(k)\| \to 1 \ \text{ and } \ x^*(k) + \sum_{i=0}^{m} x_i^*(k) \to 0. \tag{4.9}$$

Without loss of generality we assume that

$$x^*(k) \xrightarrow{w^*} x^* \text{ and } x_i^*(k) \xrightarrow{w^*} x_i^* \ (i = 0, 1, \ldots, m),$$

and hence it follows from (2.1) that

$$x_i^* \in D^*F_i(\bar{x}, \bar{y}_i)(0), \ x^* \in N(\Omega, \bar{x}), \text{ and } x^* + \sum_{i=0}^{m} x_i^* = 0.$$

Further $\|x^*\| + \sum_{i=0}^m \|x_i^*\| \neq 0$ by (4.9) and thanks to the assumption that each F_i is partially sequentially normally compact at $(\bar{x}, \bar{y}_i)$. Thus (ii) holds, and the proof is completed. ∎

As already noted, every closed multifunction between two finite dimensional spaces is partially sequentially normally compact at each point in its graph, and every closed convex cone in a finite dimensional space is dually compact. Thus, the following corollary is a consequence of Theorem 4.1.

COROLLARY 4.1. *Let Assumptions* 4.1 *and* 4.2 *hold, and suppose that* X, Y_i *are finite dimensional. Then one of* (i) *and* (ii) *in Theorem* 4.1 *holds.*

In the case when each F_i is pseudo-Lipschitz, we have the following sharp Lagrange multiplier rule.

THEOREM 4.2. *Let Assumptions* 4.1 *and* 4.2 *hold and* X, Y_i *be Asplund spaces. Suppose that each* C_i *is dually compact and that each* F_i *is pseudo-Lipschitz at* $(\bar{x}, \bar{y}_i)$. *Then there exists* $c_i^* \in C_i^+$ *such that*

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \text{ and } 0 \in \sum_{i=0}^{m} D^*F_i(\bar{x}, \bar{y}_i)(c_i^*) + N(\Omega, \bar{x}). \tag{4.10}$$

Proof. By Corollary 3.2, for any natural number k there exist $x_i(k) \in \bar{x} + \frac{1}{k}B_X$, $w(k) \in \Omega \cap (\bar{x} + \frac{1}{k}B_X)$, $y_i(k) \in F_i(x_i) \cap (\bar{y}_i + \frac{1}{k}B_{Y_i})$, and $c_i^*(k) \in C_i^+$ such that

$$\sum_{i=0}^{m} \|c_i^*(k)\| = 1 \tag{4.11}$$

and

$$0 \in \sum_{i=0}^{m} \hat{D}^*F_i(x_i(k), y_i(k)) \left(c_i^*(k) + \frac{1}{k}B_{Y_i^*}\right) \cap MB_{X^*} + \hat{N}(\Omega, w(k)) \cap MB_{X^*} + \frac{1}{k}B_{X^*},$$

where $M > 0$ is a constant independent of k. Hence there exist

$$x_i^*(k) \in \hat{D}^*F_i(x_i(k), y_i(k)) \left(c_i^*(k) + \frac{1}{k}B_{Y_i^*}\right) \text{ and } x^*(k) \in \hat{N}(\Omega, w(k))$$

such that

$$\max\{\|x^*(k)\|, \ \max\{\|x_i^*(k)\| : \ 0 \leq i \leq m\}\} \leq M \text{ and } x^*(k) + \sum_{i=0}^{m} x_i^*(k) \to 0.$$

Without loss of generality, we can assume that

$$x^*(k) \xrightarrow{w^*} x^*, \ x_i^*(k) \xrightarrow{w^*} x_i^*, \text{ and } c_i^*(k) \xrightarrow{w^*} \tilde{c}_i^* \quad \text{for } i = 0, 1, \ldots, m. \tag{4.12}$$

Hence,

$$x^* \in N(\Omega, \bar{x}), \ x_i^* \in D^*F_i(\bar{x}, \bar{y}_i)(\tilde{c}_i^*) \ (i = 0, 1, \ldots, m), \text{ and } x^* + \sum_{i=0}^{m} x_i^* = 0,$$

and so

$$0 \in \sum_{i=0}^{m} D^*F_i(\bar{x}, \bar{y}_i)(\tilde{c}_i^*) + N(\Omega, \bar{x}). \tag{4.13}$$

Since each C_i is dually compact, (4.11), (4.12), and (4.2) imply that $\sum_{i=0}^{m} \|\tilde{c}_i^*\| \neq 0$. It follows from (4.13) that (4.10) holds with $c_i^* = \frac{\tilde{c}_i^*}{\sum_{j=0}^{m} \|\tilde{c}_j^*\|}$. The proof is completed. □

Let $\bar{x}$ be a local solution of single-objective optimization problem (1.1), and suppose that each f_i is locally Lipschitz at $\bar{x}$. Let F_i and C_i be as in the proof of Theorem 3.3. Then $\bar{x}$ is a local Pareto solution of (1.3), and each F_i is pseudo-Lipschitz at $(\bar{x}, f_i(\bar{x}))$. It is routine to verify that

$$D^*F_i(\bar{x}, f_i(\bar{x}))(s) = \partial(sf_i)(\bar{x}) \ \text{ for } 0 \leq i \leq n, \quad s \geq 0,$$

and

$$D^*F_i(\bar{x}, f_i(\bar{x}))(t) = \partial(tf_i)(\bar{x}) \ \text{ for } n+1 \leq i \leq m, \quad t \in R.$$

Thus, (4.10) reduces to (1.2).

In the remainder of this section, we consider the case when X, Y_i are general Banach spaces. In this case we need the notion of the normal closedness.

We say that Ω is normally closed at $x \in \Omega$ if for (generalized) sequences

$$x_n \to x, \ x_n^* \in N_c(\Omega, x_n), \ x_n^* \xrightarrow{w^*} x^* \ \text{ implies } \ x^* \in N_c(\Omega, x)$$

(see [4, Corollary, p. 58]).

It is known that Ω is normally closed at each point of Ω if Ω is convex. Moreover, if Ω is epi-Lipschitz around $x \in \Omega$, then Ω is normally closed at x. We say that a closed multifunction $\Phi : X \to 2^Y$ is normally closed at $(x, y) \in \mathrm{Gr}(\Phi)$ if $\mathrm{Gr}(\Phi)$ is normally closed at (x, y) (see [28]).

Mimicking a corresponding notion introduced in [17], we say that $\Phi : X \to 2^Y$ is partially sequentially normally compact at $(x, y) \in \mathrm{Gr}(\Phi)$ in the Clarke sense if for any (generalized) sequence $\{(x_n, y_n, x_n^*, y_n^*)\}$ satisfying

$$x_n^* \in D_c^*\Phi(x_n, y_n)(y_n^*), \ \ (x_n, y_n) \to (x, y), \ \|y_n^*\| \to 0, \text{ and } x_n^* \xrightarrow{w^*} 0$$

one has $\|x_n^*\| \to 0$.

The following result can be proved in the same way as for Theorem 4.1 (but apply Theorem 3.2 in place of Theorem 3.1).

THEOREM 4.3. *Let Assumptions* 4.1 *and* 4.2 *hold, and suppose that each C_i is dually compact. Suppose that each F_i is partially sequentially normally compact at $(\bar{x}, \bar{y}_i)$ in the Clarke sense and that Ω and F_i are normally closed at $\bar{x}$ and $(\bar{x}, \bar{y}_i)$, respectively. Then one of the following assertions holds.*

(i) *There exist $c_i^* \in C_i^+$ such that*

$$\sum_{i=0}^{m} \|c_i^*\| = 1 \ \text{ and } \ 0 \in \sum_{i=0}^{m} D_c^*F_i(\bar{x}, \bar{y}_i)(c_i^*) + N_c(\Omega, \bar{x}).$$

(ii) *There exist $x_i^* \in D_c^*F_i(\bar{x}, \bar{y}_i)(0)$ and $w^* \in N_c(\Omega, \bar{x})$ such that*

$$\|w^*\| + \sum_{i=0}^{m} \|x_i^*\| = 1 \ \text{ and } \ w^* + \sum_{i=0}^{m} x_i^* = 0.$$

As in many classical situations, one can also provide a sufficient condition for $(\bar{x}, \bar{y}_0)$ to be a Pareto solution of (1.3), provided that a suitable convexity assumption is made.

PROPOSITION 4.1. *Let each F_i be a closed convex multifunction and Ω be a closed convex subset of X. Let $\bar{y}_0 \in F_0(\bar{x})$ and $\bar{y}_i \in F_i(\bar{x}) \cap -C_i$ for $i = 1, \ldots, m$. Assume that there exists $c_i^* \in C_i^+$ such that*

$$\langle c_0^*, c\rangle > 0 \quad \forall c \in C_0 \setminus \{0\}, \quad \sum_{i=1}^{m} \langle c_i^*, \bar{y}_i\rangle = 0 \tag{4.14}$$

and

$$0 \in \sum_{i=0}^{m} D^* F_i(\bar{x}, \bar{y}_i)(c_i^*) + N(\Omega, \bar{x}). \tag{4.15}$$

Then $(\bar{x}, \bar{y}_0)$ is a Pareto solution of the constrained multiobjective optimization problem (1.3).

Proof. By (4.15) there exists $x_i^* \in X^*$ such that

$$x_i^* \in D^* F_i(\bar{x}, \bar{y}_i)(c_i^*) \quad \text{and} \quad -\sum_{i=0}^{m} x_i^* \in N(\Omega, \bar{x}).$$

It follows from the convexity of F_i and Ω that

$$\langle x_i^*, x\rangle - \langle c_i^*, y_i\rangle \leq \langle x_i^*, \bar{x}\rangle - \langle c_i^*, \bar{y}_i\rangle \quad \forall (x, y_i) \in \mathrm{Gr}(F_i) \ \text{ and } \ i = 0, 1, \ldots, m \tag{4.16}$$

and

$$\left\langle -\sum_{i=0}^{m} x_i^*, x \right\rangle \leq \left\langle -\sum_{i=0}^{m} x_i^*, \bar{x} \right\rangle \quad \forall x \in \Omega. \tag{4.17}$$

Summing up (4.16) over all i and making use of (4.17) and (4.14) we have

$$\langle c_0^*, \bar{y}_0\rangle \leq \sum_{i=0}^{m} \langle c_i^*, y_i\rangle \ \text{ for any } x \in \Omega, \ y_i \in F_i(x), \text{ and } i = 0, 1, \ldots, m.$$

Since $c_i^* \in C_i^+$, it follows that

$$\langle c_0^*, \bar{y}_0\rangle \leq \langle c_0^*, y_0\rangle \quad \forall y_0 \in F_0 \left(\Omega \cap \bigcap_{i=1}^{m} F_i^{-1}(-C_i) \right).$$

This and the inequality in (4.14) imply that $\bar{y}_0 \in E\left(F_0\left(\Omega \cap \bigcap_{i=1}^m F_i^{-1}(-C_i)\right), C_0\right)$. The proof is completed. ∎

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