MATH 4010 | 25/02/2016

# Tutorial 5

Examples of bounded linear operator and its norm

Eg 1: Let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$

Define $T: C[0,1] \to C[0,1]$ by

$$Tu(x) = \int_0^1 K(x,y)u(y)\,dy$$

Show that $T$ is a bounded linear operator

and $\|T\| = \max_{x\in[0,1]} \int_0^1 |K(x,y)|\,dy$

Pf: Set $k(x) = \int_0^1 |K(x,y)|\,dy$. Since $K$ is continuous, so is $k$

Then $Tu \in C[0,1]$. It is obvious that $T$ is linear

Since $\|Tu\|_{C[0,1]} = \max_{x\in[0,1]} |Tu(x)| = \max_{x\in[0,1]} \left|\int_0^1 K(x,y)u(y)\,dy\right|$

$$\leq \max_{y\in[0,1]} |u(y)| \max_{x\in[0,1]} \int_0^1 |K(x,y)|\,dy = \|k\|_{C[0,1]} \|u\|_{C[0,1]}$$

Therefore $T$ is bounded and $\|T\| \leq \|k\|_{C[0,1]}$

Now, it sufficies to show that $\|T\| \geq \|k\|_{C[0,1]}$

Since $k$ is continous in $[0,1]$, then $\exists\, x_0 \in [0,1]$ s.t. $k(x_0) = \|k\|_{C[0,1]}$

Set $\varphi(y) = \operatorname{sgn} K(x_0,y) = \begin{cases} 1 & \text{if } K(x_0,y) > 0 \\ -1 & \text{if } K(x_0,y) < 0. \end{cases}$

By Lusin's Thm, $\exists\, \{u_n\} \subset C[0,1]$ (w.l.o.g. with $|u_n(y)| \leq 1$) s.t. $u_n \to \varphi$

Thus, $(\|k\|_{C[0,1]} =)\, k(x_0) = \int_0^1 K(x_0,y)\varphi(y)\,dy$

$$= \lim_{n\to\infty} \int_0^1 K(x_0,y)u_n(y)\,dy \quad \text{by L.D.C.T.}$$

$$= \lim_{n\to\infty} Tu_n(x_0) \leq \overline{\lim_{n\to\infty}} \|Tu_n\|_{C[0,1]}$$

$$\leq \overline{\lim_{n\to\infty}} \|Tu_n\|_{C[0,1]}$$

$$\leq \overline{\lim_{n\to\infty}} \|T\| \|u_n\|_{C[0,1]}$$

$$\leq \|T\|$$

Eg 2. Let $A=(a_{ij})$ $(i,j=1,2,\cdots)$ be a infinite matrix.

Define $A: x \to Ax$ by $y=Ax$ such that $y_i=\sum_j a_{ij}x_j$, $\forall x=\{x_j\}, y=\{y_i\}$

Then: (i) If $c=\sup_j \sum_i |a_{ij}|<+\infty$, then $A$ is a bounded linear operator on $\ell^1$ and $\|A\|_1 = c$

(ii) If $c=\sup_i \sum_j |a_{ij}|<+\infty$, then $A$ is a bounded linear operator on $\ell^\infty$ and $\|A\|_\infty = c$

(iii) If $c=\left(\sum(\sum|a_{ij}|^q)^{\frac{p}{q}}\right)^{\frac{1}{p}}<+\infty$ for $1<p,q<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$

then $A$ is a bounded linear operator on $\ell^p$ and $\|A\|_p \leq c$.

Pf: (i) Since $\sum_j |a_{ij}x_j| \leq \sup_j |a_{ij}| \sum_j |x_j|$

$\sup_j |a_{ij}| \leq c < +\infty$

$\sum_j |x_j| < +\infty$, since $\{x_j\} \in \ell^1$

then $\sum_j |a_{ij}x_j| < +\infty$

i.e. $y_i$ is well-defined.

$$\|Ax\| = \|y_i\|_{\ell^1} = \sum_i |y_i| = \sum_i |\sum_j a_{ij}x_j| \leq \sum_i \sup_j |a_{ij}| \sum_j |x_j| \leq c\|x\|_{\ell^1}$$

Hence, $A$ is bounded and $\|A\|_1 \leq c$.

It sufficies to show $\|A\|_1 \geq c$, i.e. $\forall c'<c$, $\|A\|_1 > c'$.

Since $c=\sup_j \sum_i |a_{ij}|<+\infty$, then $\forall c'<c$, $\exists j_0$ s.t. $c<\sum_i |a_{ij_0}|$

Let $e_j=(0,\cdots,0,\underset{\substack{\uparrow \\ j\text{-th}}}{1},0,\cdots)$ be a basis of $\ell^1$

then $Ae_j = (a_{1j}, a_{2j}, \cdots)$

$$c' < \sum_i |a_{ij_0}| = \|Ae_{j_0}\| \leq \|A\|_1 \|e_{j_0}\| = \|A\|_1.$$

(ii) $\sum_j |a_{ij}x_j| \leq \sup_j |x_j| \sum_j |a_{ij}| \leq c\|x\|_{\ell^\infty} < +\infty$, $\forall x \in \ell^\infty$.

So, $y_i$ is well-defined and moreover

$$\|Ax\|_{\ell^\infty} = \|y\|_{\ell^\infty} = \sup_i |y_i| \leq \sup_j |x_j| \sup_i \sum_j |a_{ij}| \leq c\|x\|_{\ell^\infty}$$

Hence $A$ is bounded and $\|A\|_\infty \leq c$

It sufficies to show $\|A\|_\infty \geq c$, i.e. $\forall c'<c$, $\|A\|_\infty > c'$.

Since $\forall c'<c=\sup_i \sum_j |a_{ij}|<+\infty$, $\exists i_0$ s.t. $c'<\sum_j |a_{i_0 j}|$

Set $x_j = \operatorname{sgn} a_{i_0 j}$, then $\|x\|_{\ell^\infty}=1<+\infty$ and

$$c' < \sum_j |a_{i_0 j}| = \sum_j a_{i_0 j} x_j = y_{i_0} \leq \|A\|_\infty \|x\|_{\ell^\infty} = \|A\|_\infty$$

(iii) By Hölder inequality

$$|y_i| \leq \left(\sum_j |a_{ij}|^q\right)^{\frac{1}{q}} \left(\sum_j |x_j|^p\right)^{\frac{1}{p}} < +\infty, \ \forall x \in \ell^p.$$

Then $A$ is well-defined

$$\|Ax\|_{\ell^p}^p = \|y\|_{\ell^p}^p = \sum_i |y_i|^p = \sum_i \left|\sum_j a_{ij} x_j\right|^p \leq \sum_i \left(\sum_j |a_{ij}|^q\right)^{\frac{p}{q}} \|x\|_{\ell^p}^p$$

i.e. $\|y\|_{\ell^p} \leq \left(\sum_i \left(\sum_j |a_{ij}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \|x\|_{\ell^p}^p$

Hence $A$ is bounded and $\|A\| \leq C$.