General Information

- Textbook: *Introduction to Real Analysis* by Robert G. Bartle, Donald R. Sherbert. (Try to google the title of the textbook for MORE information!)
- The course MATH2060 Mathematical Analysis II will also use this textbook.
- I am the tutor of this section. You may call me **Ernest**. My office is located at **LSB G06** and my office hour for this course is **Thursday 10:30-12:30**. You may come to me during this session if you need any help. My email address is **ylfan@math.cuhk.edu.hk**. You are welcomed to send me an email if you cannot find me.
- Please visit the course web-page at `https://www.math.cuhk.edu.hk/course/1920/math2050a` frequently to get the most updated information. It shall contain the information for the Homework and Quizzes, as well as lecture notes and tutorial notes.

Review on Week 1

As an introduction to the real number system $\mathbb{R}$, we need some definitions.

Upper/Lower Bounds, Supremum/Infimum

Definition (c.f. Definition 2.3.1)**.** Let X be a **non-empty** subset of $\mathbb{R}$.

- A number $u \in \mathbb{R}$ is said to be an *upper bound* of X if $x \leq u$ for all $x \in X$.
- A number $l \in \mathbb{R}$ is said to be a *lower bound* of X if $x \geq l$ for all $x \in X$.
- X is said to be *bounded above* if it has an upper bound.
- X is said to be *bounded below* if it has a lower bound.
- X is said to be *bounded* if it is both bounded above and bounded below.
- X is said to be unbounded if it is not bounded.

Definition (c.f. Definition 2.3.2)**.** Let X be a **non-empty** subset of $\mathbb{R}$.

- The *supremum* of X, denoted by $\sup X$, is defined as the least upper bound of X. i.e. $\sup X \geq x$ for all $x \in X$ and $\sup X \leq u$ whenever u is an upper bound of X.
- The *infimum* of X, denoted by $\inf X$, is defined as the greatest lower bound of X. i.e. $\inf X \leq x$ for all $x \in X$ and $\inf X \geq l$ whenever l is a lower bound of X.

The following lemma is also useful to determine whether an upper bound u of a non-empty subset X of $\mathbb{R}$ is a supremum. (Can you formulate a lemma corresponding to the case of infimum?)

Lemma (c.f. Lemma 2.3.3 and Lemma 2.3.4). *Let u be an upper bound of a non-empty subset X of $\mathbb{R}$. The following statements are equivalent:*

(i) u is the supremum of X, i.e. $u = \sup X$.

(ii) If $v < u$, then there exists $x \in X$ such that $v < x$.

(iii) For every $\varepsilon > 0$, there exists $x \in X$ such that $u - \varepsilon < x$.

The Completeness Property

The most important property of the real number system $\mathbb{R}$ is the following, which we called the **the completeness property** or **the axiom of completeness**.

The Completeness Property of $\mathbb{R}$ (c.f. 2.3.6). *Every bounded above non-empty subset of $\mathbb{R}$ has a supremum in $\mathbb{R}$.*

One of its application is to show the **Archimedean Property**, which states that the set of natural numbers is unbounded.

Archimedean Property (c.f. 2.4.3). *If $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x \leq n$.*

This yields the following useful corollary (Especially when we do exercises!):

Corollary (c.f. Corollary 2.4.5). *If $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$.*

Exercises

Question 1. Determine the supremum and infimum of the following sets (if such exist):

(a) $X_1 = (0, 1]$

(b) $X_2 = \mathbb{N}$

(c) $X_3 = [0, 1) \cup \{2\}$

(d) $X_4 = (0, 1) \cap \mathbb{Q}$

(e) $X_5 = \mathbb{R} \setminus [-1, 1]$

(f) $X_6 = \{n + 1/n : n \in \mathbb{N}\}$

Solution. As a warm up exercise, no proofs are needed. Try to visualize the given sets.

(a) $\inf X_1 = 0$; $\sup X_1 = 1$.

(b) $\inf X_2 = 1$; $\sup X_2$ does not exist.

(c) $\inf X_3 = 0$; $\sup X_3 = 2$.

(d) $\inf X_4 = 0$; $\sup X_4 = 1$.

(e) $\inf X_5$ does not exist; $\sup X_5$ does not exist.

(f) $\inf X_6 = 2$; $\sup X_6$ does not exist.

Question 2 (c.f. Section 2.4, Ex.1)**.** Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.

Solution. We need to show that (i) 1 is an upper bound of the set and (ii) if u is an upper bound, then $1 \leq u$.

To show that $1 - 1/n \leq 1$ for all $n \in \mathbb{N}$, let $n \in \mathbb{N}$. Note that $1/n \geq 0$. hence $1 - 1/n \leq 1 - 0 = 1$.

To show that 1 is the least upper bound, suppose on a contrary that there is an upper bound $u \in \mathbb{R}$ of the set $\{1-1/n : n \in \mathbb{N}\}$ such that $u < 1$. Since $1-u > 0$, by **Archimedean Property** (Corollary 2.4.5), there exists $n \in \mathbb{N}$ such that $0 < 1/n < 1 - u$. It follows that

$$u < 1 - \frac{1}{n},$$

contradict the face that u is an upper bound. Therefore 1 must be the least upper bound.

Remark. Homework 1: Section 2.4, Q2 is similar.

Question 3 (c.f. Section 2.4, Ex.7)**.** Let A, B be bounded non-empty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

Solution. I shall prove the case of infimum and leave case of supremum as an exercise, the arguments are similar. Also, note that the equality can be shown by showing the "$\geq$" case and the "$\leq$" case.

To show $\inf(A+B) \geq \inf A + \inf B$, let $a \in A$ and $b \in B$. Since $\inf A$ and $\inf B$ are lower bounds of A and B respectively. Hence $a \geq \inf A$ and $b \geq \inf B$. Therefore

$$a + b \geq \inf A + \inf B.$$

Since $a \in A$ and $b \in B$ are arbitrary, $\inf A + \inf B$ is a lower bound of $A + B$. It follows that $\inf(A + B) \geq \inf A + \inf B$.

To show $\inf(A + B) \leq \inf A + \inf B$, let $a \in A$ and $b \in B$. Since $\inf(A + B)$ is a lower bound of $A + B$, we have $\inf(A + B) \leq a + b$. Then

$$\inf(A + B) - a \leq b.$$

Since $b \in B$ is arbitrary, $\inf(A + B) - a$ is a lower bound of B. Therefore

$$\inf(A + B) - a \leq \inf B.$$

We now have

$$\inf(A + B) - \inf B \leq a.$$

Since $a \in A$ is arbitrary, $\inf(A + B) - \inf B$ is a lower bound of A. Therefore

$$\inf(A + B) - \inf B \leq \inf A.$$

Finally we arrived at the desired conclusion

$$\inf(A + B) \leq \inf A + \inf B.$$

Remark. Homework 1: Section 2.4, Q4 is similar.