THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 6: Section 4.1: 11, 12c, d, 15

11. Use the definition of limit to prove the following. (2 marks each)

(a) $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$, (b) $\lim_{x\to 6} \frac{x^2-3x}{x+3} = 2$.

Solution.

(a) Note that if $|x-3| < \frac{1}{2}$, then $\frac{5}{2} < x < \frac{7}{2}$ and hence $1 < 4x-9$, which means that when $|x-3| < \frac{1}{2}$, the distance between the denominator $4x-9$ and 0 will be at least 1. Note also

$$\begin{aligned}\left|\frac{2x+3}{4x-9} - 3\right| &= \left|\frac{-10x+30}{4x-9}\right| \\ &= \frac{10}{|4x-9|}\,|x-3| \\ &\leq 10|x-3|, \qquad\qquad \text{when } |x-3| < \frac{1}{2}\end{aligned}$$

Let $\epsilon > 0$. From above, if we put $\delta = \min(\frac{1}{2}, \frac{\epsilon}{10})$, then

$$\left|\frac{2x+3}{4x-9} - 3\right| < \epsilon, \quad \text{when } 0 < |x-3| < \delta$$

This shows $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

(b) Note that if $|x-6| < 1$, then $5 < x < 7$. In particular, we have $8 < x+3$ and $|x+1| < 8$. Now, when $|x-6| < 1$,

$$\begin{aligned}\left|\frac{x^2-3x}{x+3} - 2\right| &= \left|\frac{x^2-5x-6}{x+3}\right| \\ &= \left|\frac{(x+1)(x-6)}{x+3}\right| \\ &\leq \frac{|x+1|}{8}|x-6| \\ &\leq |x-6|\end{aligned}$$

Let $\epsilon > 0$. If we put $\delta = \min(1, \epsilon)$, then

$$\left|\frac{x^2-3x}{x+3} - 2\right| < \epsilon, \quad \text{when } 0 < |x-6| < \delta$$

By definition, we have $\lim_{x\to 6} \frac{x^2-3x}{x+3} = 2$.

12. Show that the following limits do not exist. (1.5 marks each)

(c) $\lim_{x\to 0}(x+\operatorname{sgn}(x))$, (d) $\lim_{x\to 0}\sin(1/x^2)$.

Solution.

(c) For $x > 0$, we have $x + \operatorname{sgn}(x) = x + 1 > 1$.
For $x < 0$, we have $x + \operatorname{sgn}(x) = x - 1 < -1$.
Therefore,

$$\begin{array}{ll} \lim_{x\to 0^+}(x+\operatorname{sgn}(x)) \geq 1 & \text{whenever exists} \\ \lim_{x\to 0^-}(x+\operatorname{sgn}(x)) \leq -1 & \text{whenever exists} \end{array}$$

We can conclude that $\lim_{x\to 0}(x+\operatorname{sgn}(x))$ does not exist.

(d) Let $f(x) = \sin(1/x^2)$.
Consider two sequences $(x_n), (y_n)$, where $x_n = \dfrac{1}{\sqrt{2\pi n}}$ and $y_n = \dfrac{1}{\sqrt{\frac{\pi}{2}+2\pi n}}$.
Notice that

(i) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$

(ii) $x_n \neq 0$, $y_n \neq 0$ for all $n \in \mathbb{N}$

Moreover,

$$\begin{array}{l} \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\sin(2\pi n) = 0 \\ \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}\sin(\frac{\pi}{2}+2\pi n) = 1 \end{array}$$

By Divergence criteria, $\lim_{x\to 0}\sin(1/x^2)$ does not exist.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$. (1.5 marks)

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c. (1.5 marks)

Solution.

(a) Let $\epsilon > 0$. We put $\delta = \epsilon$. If $0 < |x - 0| < \delta$, then

$$\begin{array}{ll} |f(x) - 0| = |x| < \epsilon & \text{if } x \text{ is rational} \\ |f(x) - 0| = 0 < \epsilon & \text{if } x \text{ is irrational} \end{array}$$

Since $|f(x) - 0| < \epsilon$ whenever $0 < |x - 0| < \delta$, we have $\lim_{x\to 0} f(x) = 0$.

(b) By the density theorem, we can pick two sequences $(x_n), (y_n)$ such that for all $n \in \mathbb{N}$

(i) $x_n \in \mathbb{Q}$, $y_n \in \mathbb{R}\setminus\mathbb{Q}$,

(ii) $x_n, y_n \in (c, c + \frac{1}{n})$

Note that (ii) tells us that $x_n, y_n \neq c$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$. Now, (i) shows that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} f(y_n) = 0$ Therefore, $\lim_{x\to c} f(x)$ does not exist when $c \neq 0$.