THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 3: p.70: 18; p.77: 2,7

18. let $X = (x_n)$ be a sequence of positive real numbers such that $\lim_{n\to\infty}(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.

(4 marks)

Solution.

First, we would show that there is some number $r > 1$ such that $\frac{x_{n+1}}{x_n} > r$ when n is sufficiently large: since $\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = L > 1$, if we put $\epsilon = \frac{L-1}{2} > 0$, there is some $N \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \frac{L-1}{2} \text{ for every } n \geq N.$$

In particular, we would have $\frac{x_{n+1}}{x_n} > L - \left(\frac{L-1}{2}\right) = \frac{L+1}{2}$. The number r we proposed is $\frac{L+1}{2}$ and what we mean by sufficiently large n is $n \geq N$.

Second, for any $n \geq N$, we have the inequality that

$$x_{n+1} > rx_n > \cdots > r^{n-N+1}x_N = r^n\left(\frac{x_N}{r^{N-1}}\right),$$

because (x_n) is a sequence of positive numbers. Note that $\frac{x_N}{r^{N-1}}$ is a positive constant, while r^n is an unbounded sequence. We can conclude that X is also unbounded.

2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

(3 marks)

Solution.

First we show that $x_n > 1$ for every $n \in \mathbb{N}$. It is given that $x_1 > 1$. Assume $x_k > 1$ for some positive integer k. We have $x_k > 1 \iff 0 < \frac{1}{x_k} < 1 \iff 1 < 2 - \frac{1}{x_k} < 2$. That is, $1 < x_{k+1} < 2$. By induction, we have shown that $x_n > 1$ for every $n \in \mathbb{N}$. As we shall see in the proof of induction that $1 < x_{n+1} < 2$ for every $n \in \mathbb{N}$. Thus, the sequence (x_n) is bounded.

Second, we show that (x_n) is monotone. Note that

$$x_{n+2} - x_{n+1} = (2 - \frac{1}{x_{n+1}}) - (2 - \frac{1}{x_n}) = \frac{1}{x_n} - \frac{1}{x_{n+1}} = \frac{x_{n+1} - x_n}{x_n x_{n+1}}.$$

We have shown that the denominator $x_n x_{n+1}$ is a positive number, hence $x_{n+2} \geq x_{n+1}$ if and only if $x_{n+1} \geq x_n$. The sequence is therefore monotone. It may be increasing or decreasing, depending on the sign of $x_2 - x_1$.

By the Monotone Convergence theorem, $\lim\limits_{n\to\infty} x_n$ exists and we denote it by L, where $1 \leq L \leq 2$. From the recursive formula, we see that $L = 2 - \dfrac{1}{L}$, which is equivalent to the equation $(L-1)^2 = 0$. We find that the limit of (x_n) is 1.

7. Let $x_1 := a > 0$ and $x_{n+1} = x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.

(3 marks)

Solution.

It is easy to see that $x_n > 0$ for every $n \in \mathbb{N}$: it is given that $x_1 > 0$. Assume that $x_k > 0$ for some positive integer k. Then, $x_{k+1} = x_k + 1/x_k$ is the sum of two positive number and is still positive. By induction, we have shown the claim. Moreover, from which we see that (x_n) is a strictly increasing sequence.

Now, we will show that the sequence diverges. Suppose not, let $L = \lim\limits_{n\to\infty} x_n$. Since (x_n) is increasing, we have $L \geq x_1 > 0$ and thus $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{L}$. By the recursive formula, we have

$$L = L + \frac{1}{L}, \text{ which implies } \frac{1}{L} = 0.$$

However, every multiplicative inverse is nonzero. Therefore, the sequence diverges.