MATH2050A,B: Analysis I: Complementary Exercise
December 2019

1. Let $A \subset \mathbb{R}$ be bounded above but not below. Suppose A has the property that

$$(*) \quad \text{if } a_1 < x < a_2 \ \text{ and } \ a_1, a_2 \in A \ \text{ then } \ x \in A.$$

   Show that A is an interval.

2. Let $a > 0$, and

$$\begin{aligned} A :=& \{x > 0 : a \leqslant x^2\}; \\ f(x) :=& \frac{1}{2}(x + \frac{a}{x}) \quad \forall x \in A. \end{aligned}$$

   Let $x_1 \in A$ and $x_{n+1} := f(x_n)$ for all nature number n. Show that $f(x) \in A$ and $f(x) \leqslant x$ for all $x \in A$, and further that the sequence (x_n) converges with limit $z = \sqrt{a}$, that is $z > 0$ and $z^2 = a$.

   (You may use the Monotone Convergence Theorem for sequences and the computation rules for limits)

3. (a) Use $\varepsilon - \delta$ terminology to show for $c > 0$ that

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}.$$

   (b) Compute the limit

$$\lim_{x \to 0} \frac{\sqrt{1 + 3x} - \sqrt{1 + 2x}}{x + 2x^2}.$$

4. Use $\varepsilon - \delta$ terminology show that

   (a) $\displaystyle\lim_{x\to 1} \frac{x^2+2}{x^2-2} = -3.$

   (b) Let f_1, f_2 be real-valued functions on $A \subset \mathbb{R}$ and x_0 be a cluster point with respect to A such that $\lim_{x\to x_0} f_i(x) = \ell_i \in \mathbb{R}$ $(i = 1, 2)$. Suppose that $f_2(x) \neq 0$ for all $x \in A$. Show that $\displaystyle\lim_{x\to x_0} \frac{f_1(x)}{f_2(x)} = \frac{\ell_1}{\ell_2}.$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function (say with period $p : f(x+p) = f(x)\ \forall x \in \mathbb{R}$). Show that

   (a) f attains its maximum value.

   (b) f is uniformly continuous on $\mathbb{R}$.
   (You may apply the Bolzano-Weierstrass Theorem).

6. Let f be a $\mathbb{R}$-valued function defined on $\mathbb{R}$ and let A be a countable dense subset of $\mathbb{R}$. Assume that the function f is continuous on A. Put

$$D := \{x \in \mathbb{R} \setminus A : f \text{ is conntinuous at } x\}.$$

   Prove or disprove the following cases.

   (i) The set D is nonempty.
   (ii) The set D is dense in $\mathbb{R}$.

7. Let f be a continuous $\mathbb{R}$-valued function defined on $\mathbb{R}$. Assume that the limits $L' := \lim_{x\to-\infty} f(x)$ and $L := \lim_{x\to+\infty} f(x)$ both exist. Consider the following cases:

$$L' < L \quad ; \quad L' > L \quad \text{and } L = L'.$$

   Prove or disprove the following statements for the above cases:

   (i) f attains at least one of its maximal values or minimum value.
   (ii) f attains its maximal values and its minimum value.
   (iii f is uniformly continuous on $\mathbb{R}$.