Tutorial 3

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1. Stability for diffusion equation by maximum principle.(Stability in 'uniform' sense)

Thoerem: Soppose $u_i(x,t), i = 1,2$ are solutions of the following Initial-Boundary-Value-Problem:

$$\partial_t u_i = k\partial_x^2 u_i \quad 0 \le x \le l, t \ge 0$$

$$u_i(x, t = 0) = \phi_i(x) \quad 0 \le x \le l$$

$$u_i(x = 0, t) = 0,\ u_i(x = l, t) = 0$$

Then

$$\max_R |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|$$

Proof: Set $v(x,t) = u_1(x,t) - u_2(x,t)$, the v satisfys

$$\partial_t v(x,t) = k\partial_x^2 v(x,t)$$

$$v(x, t = 0) = \phi_1(x) - \phi_2(x)$$

$$v(x = 0, t) = 0,\ v(x = l, t) = 0$$

Apply Maximum Principle to v, we have

$$\max_R v(x,t) \le \max_{\partial_p R} v(x,t) \le \max\{0, \max_{0\le x\le l} \phi_1(x) - \phi_2(x)\} \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|$$

Apply Minmum Principle to v, we have

$$\min_R v(x,t) \ge \min_{\partial_p R} v(x,t) \ge \min\{0, \min_{0\le x\le l} \phi_1(x) - \phi_2(x)\}$$

that is

$$\max_R -v(x,t) \le -\min\{0, \min_{0\le x\le l} \phi_1(x) - \phi_2(x)\} = \max\{0, \max_{0\le x\le l} -(\phi_1(x) - \phi_2(x))\} \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|$$

Then

$$\max_R |v(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|$$

Thus

$$\max_R |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|$$

2. Error function of statistics on P50

Fact:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof of the fact:

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+x^2)} dxdy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r d\theta dr = \pi$$

hence $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ $\square$

Define

$$\mathcal{E}rf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}dp$$

thus $\mathcal{E}rf(0) = 0$, and $\mathcal{E}rf(\infty) = \lim_{x\to\infty}\mathcal{E}rf(x) = 1$.

Example 1:

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp = \frac{1}{2} + \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})$$

Example 2: Consider the diffusion eqution with the initial data $\phi(x) = e^{-x}$, thus the solution is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}e^{-y}dy$$

$$= \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2-2xy+4kty}{4kt}}dy = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}}e^{kt-x}dy$$

$$= e^{kt-x}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\frac{y+2kt-x}{\sqrt{4kt}})^2}d(\frac{y+2kt-x}{\sqrt{4kt}}) = e^{kt-x}$$

3. Exercise 8 on P45

Consider the diffusion equation on $(0,l)$ with the Robin boundary conditions $u_x(0,t) - a_0u(0,t) = 0$ and $u_x(l,t) + a_lu(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*".)

Answer: For the diffusion equation $u_t - ku_{xx} = 0$,

$$\frac{d}{dt}\int_0^l u^2dx = \int_0^l 2uu_tdx = \int_0^l 2kuu_{xx}dx = 2kuu_x\Big|_0^l - \int_0^l 2ku_x^2dx$$

$$= 2k[u(l,t)u_x(l,t) - u(0,t)u_x(0,t)] - \int_0^l 2ku_x^2dx.$$

Then the Robin boundary conditions imply

$$\frac{d}{dt}\int_0^l u^2dx = -2k[a_lu(l,t)^2 + a_0u(0,t)^2] - \int_0^l 2ku_x^2dx \le 0,$$

where $-2k[a_lu(l,t)^2+a_0u(0,t)^2] \le 0$ shows that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)dx$. $\square$

4. Exercise 8 on P51

Show taht for any fixed $\delta > 0$(no matter how small),

$$\max_{\delta\le|x|<\infty} S(x,t) \to 0 \qquad \text{as } t \to 0.$$

[This means that thee tail of $S(x,t)$ is "uniformly small".]

Answer: By the definition of $S(x,t)$,

$$\max_{\delta\le|x|<\infty} S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt},$$

so

$$\lim_{t\to 0^+}\max_{\delta\le|x|<\infty} S(x,t) = \lim_{t\to 0^+}\frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt} = \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{4\pi k}}e^{-x\delta^2/4k} = 0. \quad \square$$