Tutorial 1

January 21,2016

1.Second order ODE(continued)

$$F(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}) = 0$$

where F is a given function.

We consider the simplist second order ODE, constant coefficients second order linear ODE:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

where $a \neq 0, b, c$ are constants and $f(t)$ is a given function.If $f(t) = 0$, it is called homogeneous equation. If $f(t) \neq 0$, it is called inhomogeneous equation.

How to solve this equation?

Firstly, find the general solution of the homogenous case. The corresponding charicteristic equation is

$$ar^2 + br + c = 0$$

Then we have $r = r_1, r = r_2$.If $r_1 \neq r_2$ are real, then the general solution of the homogenous equation is

$$x_h(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$

If $r_1 = r_2 = r$ is real, then the general solution of the homogenous equation is

$$x_h(t) = C_1e^{rt} + C_2te^{rt}.$$

If $r = \alpha \pm i\beta$, then the general solution of the homogenous equation is

$$x_h(t) = C_1e^{\alpha t} \cos \beta t + C_2e^{\alpha t} \sin \beta t.$$

where C_1 and C_2 are arbitrary constants.

Secondly, find a particular solution of the inhomogeneous equation. If $f(t)$ is of the form $P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$, we can assume that a particular solution is of the form

$$x_p(t) = t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$$

where s is the number of times 0 is a root of the characteristic equation.
If $f(t)$ is of the form $P_n(t)e^{\alpha t}$,we can assume that a particular solution is of the form

$$t^se^{\alpha t}(A_0t^n + A_1t^{n-1} + \cdots + A_n)$$

where s is the number of times α is a root of the characteristic equation.
If $f(t)$ is of the form $P_n(t)e^{\alpha t}\cos\beta t$ or $P_n(t)e^{\alpha t}\sin\beta t$, then we can find a particular solution of the form

$$t^s e^{\alpha t}[(A_0t^n + A_1t^{n-1} + \cdots + A_n)\cos\beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)\sin\beta t]$$

where s is the number of times $\alpha + i\beta$ is a root of the characteristic equation. $A_0, \cdots, A_n$ and $B_0, \cdots, B_n$ are undetermined coefficients.

Hence, the general solution of the inhomogeneous equation is

$$x(t) = x_h(t) + x_p(t)$$

Remark: The above method is called Method of Undetermined Coefficients. There are other methods ,such as Variation of Parameters. You can take a reference of Boyce's book or any other ODE books.

2.Schrodinger Equation (Example 7 on P17)

Consider the Hydrogem Atom. This is an electron moving around a proton.Let m be the mass of the electron, e the charge, and h Planck's constant divided by 2π.Let the origin of coordinates (x, y, z) be the position of the proton and let $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ be the spherial coordinate.

Let $u(x, y, z, t)$ be the wave function which represents a possible state of the electron, and $|u|^2$ represents the probability density of the electron at position x and time t. If D is any region of the space , then $\iiint_D |u|^2 dxdydz$ is the probability of finding the electron in the region D at time t. Thus

$$\iiint_{\mathbb{R}^n} |u|^2 dxdydz = 1$$

The motion of the electron satisfies Schrodinger equation:

$$-ihu_t = \frac{h^2}{2m}\triangle u + \frac{e^2}{r}u$$

in all of space $-\infty < x, y, z < \infty$

Remark:

1.The coefficient $\frac{e^2}{r}$ is called the potential. For any othe atom with a single electron, e^2 is replaced by Ze^2, where Z is the atomic number.

2.With many particles(electrons), the wave function u is a function of a large number of variables. The Shrodinger Equation then becomes:

$$-ihu_t = \sum_{n=1}^{n} \frac{h^2}{2m_i}(u_{x_ix_i} + u_{y_iy_i} + u_{z_iz_i}) + V(x_1, \cdots, z_n)u$$

where the potential V depends onf all the $3n$ coordinates.

3.If we use the operator A to denote the observable quantities, then the expected value of the observable A equals

$$\iiint_D Au(x, y, z, t) \cdot \overline{u}(x, y, z, t)dxdydz$$

For example, the position is given by the operator $Au = xu$, and the momentum is given by $Au = -ih\nabla u$.

3. Newton's law of cooling for heat equation.

Consider a uniform rod with length l which is insulated along $0 \le x \le l$.Let $u(x,t)$ be the temperature at position x and time t. We have know that $u(x,t)$ satisfies

$$\partial_t u - c^2 \partial_x^2 u = 0, \quad 0 < x < l, \quad t \ge 0$$

If the end $x = l$ is immersed in the reservior of temperature $g(t)$ and the heat were exchanged between the rod and the reservior to obey Newton's law of cooling which says that the rate of heat loss is propotional to the difference in temperature between the body and its surroudings.
That is,

$$\partial_x u(l,t) = -a(u(l,t) - g(t))$$

where a is positive and the negative sign $-$ means that if the rod is hot and the reservior is cold, then $\partial_x u(l,t) < 0$ and the temperature difference $u(l,t) - g(t) > 0$ thus there should be a $-$. This is an inhomogeneous Robin boundary condition.

4.Condition at infinity.

When the domain is unbounded, what kind of the condition should we impose? The physics usually provides conditions at infinity.

For example, consider the Schrodinger equation where the domain is the whole space. Since

$$\iiint_{\mathbb{R}^n} |u|^2 dxdydz = 1$$

we know that u "vanishes at infinity", that is,

$$\lim_{r \to +\infty} u(x,y,z,t) = 0$$

where r is the spherical coordinates.

For example,consider a point light in the whole space. The light wave is radiating from the light outward to infinity. In this case, the energy must scatter to infinity and no energy is radiating from the infinity to the light source.The appropriate condition at infinity is "sommerfeld's outgoing radiation condition":

$$\lim_{r \to +\infty} r\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) = 0$$

where r is the spherical coordinates.