Chapter 3. Systems of Hyperbolic Conservation Laws and Glimm Scheme

§3.1 Introduction

We consider a general system of n equations in one space dimension

$$\begin{cases} u_t + f(u)_x = 0, \quad x \in R,\ t > 0,\ u \in R^n,\ f \in R^n,\ f \in C^2 \\ u(x, t = 0) = u_0(x) \end{cases} \tag{3.1}$$

In this chapter, we will discuss the following five main topics:

- Riemann problem for systems of conservation laws (P. Lax)
- Wave interaction estimates
- Glimm Scheme and Glimm's functional
- Convergence of Glimm's method (Random choice method)
- Uniqueness of Glimm's solution (A. Bressen)

Glimm scheme is very important in solving the Cauchy problem. It provides a new idea and a new approach to the nonlinear partial differential equations.

In solving the Riemann problems for scalar case, we obtain two kinds of basic nonlinear waves, shock waves and centered rarefaction waves. But how can we adopt these basic waves into systems? Lax and Glimm observed that Riemann problem is not only important for scalar conservation laws, but also for systems, and they provide the building blocks for systems of conservation laws.

Before we go to the main parts, we introduce some general concepts of conservation laws.

Definition 3.1 The system (3.1) is called hyperbolic if $(\nabla_u f(u))_{n\times n}$ has only real eigenvalues, namely $\lambda_1(u) \leq \cdots\cdots \leq \lambda_n(u)$. It is called strictly hyperbolic if all eigenvalues are distinct, i.e. $\lambda_1(u) < \cdots\cdots < \lambda_n(u)$.

Nonstrictly hyperbolic cases arise from some material science and they are much complicated than the strictly hyperbolic cases, also the Glimm scheme does not work very well there. From now on, we will assume (3.1) is always strictly hyperbolic. Then we can find the corresponding right and left eigenvectors

$$r_1(u),\ r_2(u), \cdots, r_n(u),$$

$$l_1(u),\ l_2(u), \cdots, l_n(u),$$

$$\nabla f(u) \cdot r_i(u) = \lambda_i(u)\, r_i(u) \quad \text{and} \quad l_i(u) \cdot \nabla f(u) = \lambda_i(u)\, l_i(u).$$

And we denote by $R(u) = (r_1(u), \cdots, r_n(u))$ the $n \times n$ matrix of right eigenvectors, $L(u) = (l_1(u), \cdots, l_n(u))^t$ the $n \times n$ matrix of left eigenvectors. We normalize those eigenvectors such that

$$L(u) \cdot \nabla f(u) \cdot R(u) = \Lambda(u) = \mathrm{diag}\,\{\lambda_1(u), \cdots\cdots, \lambda_n(u)\}$$

and

$$L(u) \cdot R(u) = I_{n \times n}.$$

The concept of characteristic fields are very important. Consider, for example, the movement of a elastic string, which is modelled by the second order hyperbolic wave equation. And we know the sound wave propagates in two different directions. On each characteristic direction, it acts like the solution to the scalar equation. This reminds us that we can decompose the problem into simpler one by characteristic field. For each field, Lax propose the following concept.

Definition 3.2 The i-th characteristic field is genuinely nonlinear if

$$\nabla\lambda_i(u)\cdot r_i(u)\neq 0, \quad \text{for all} \quad u\,\epsilon\,\Omega\,\subset\, R^n.$$

Otherwise, if $\nabla\lambda_i(u)\cdot r_i(u)=0 \quad \text{for all} \quad u\,\epsilon\,\Omega\,\subset\, R^n$, then the i-th characteristic field is said to be linearly degenerate.

Example 1 $n=1$. Then $f^{'}(u)$ is a scalar, and $\lambda_1(u)=f^{'}(u)$, $r_1=1$. Hence $\partial_u\,\lambda_1(u)=f^{''}(u)$. Then when f is convex, it is genuinely nonlinear, and when $f(u)=\lambda u+c$ $(\lambda, c$ are constants), $f^{''}(u)\equiv 0$, it is linearly degenerate.

Example 2 p-system:

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p(v) = 0, \quad t > 0, \quad x \epsilon R, \end{cases} \tag{3.2}$$

where $p^{'} < 0, \quad p^{''} > 0$. Here we let

$$U = (v, u), \quad F(U) = (-u, p(v)),$$

then (3.2) can be written as

$$U_t + F(U)_x = 0,$$

and the Jacobian matrix is

$$dF = \begin{pmatrix} 0 & -1 \\ p^{'}(v) & 0 \end{pmatrix}$$

it has real and distinct eigenvalues

$$\lambda_1 = -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} = \lambda_2.$$

The right eigenvector corresponding to, say λ_2, is
$r_2 = \left(-1, \sqrt{-p'(v)}\right)^t$.

Then

$$\nabla\lambda_2 \cdot r_2 = \left(\frac{-p''(v)}{2\sqrt{-p'(v)}}, 0\right) \cdot \left(-1, \sqrt{-p'(v)}\right)^t = \frac{p''(v)}{2\sqrt{-p'(v)}} > 0.$$

Hence the second characteristic family is genuinely nonlinear. And in a similar way, the first family is also genuinely nonlinear.

Example 3 Consider the full gas dynamics system in Eulerian coordinates

$$\begin{aligned} &\rho_t + (\rho\, u)_x = 0, \\ &u_t + u\, u_x + p_x/\rho = 0, \\ &s_t + u\, s_x = 0, \end{aligned} \quad \begin{pmatrix} \rho \\ u \\ s \end{pmatrix} = \begin{pmatrix} \text{density} \\ \text{velocity} \\ \text{entropy} \end{pmatrix},$$

where $p = p(\rho, s), p_\rho > 0$. We denote the sound speed c by $c = \sqrt{p_\rho}$. The matrix

$$\begin{pmatrix} u & \rho & 0 \\ p_\rho/\rho & u & p_s/\rho \\ 0 & 0 & u \end{pmatrix}$$

has eigenvalues $\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c$, with corresponding right eigenvectors
$r_1 = (\rho, -c, 0)^t, r_2 = (p_s, 0, -p_\rho)^t$, and $r_3 = (\rho, c, 0)^t$.

Now we see that

$$
\begin{aligned}
\nabla \lambda_1 \cdot r_1 &= (-c_\rho, 1, -c_s) \cdot (\rho, -c, 0)^t &&= -\rho\, c_\rho - c \neq 0, \\
\nabla \lambda_2 \cdot r_2 &= (0, 1, 0) \cdot (p_s, 0, -p_\rho)^t &&= 0, \\
\nabla \lambda_3 \cdot r_3 &= (c_\rho, 1, c_s) \cdot (\rho, c, 0)^t &&= \rho\, c_\rho + c \neq 0.
\end{aligned}
$$

Thus $\lambda_1 < \lambda_2 < \lambda_3$ and λ_1 & λ_3 are genuinely nonlinear, λ_2 is linearly degenerate. The 2-nd family is the so-called entropy wave family.

Now we want to give definition of three elementary waves, namely, shock waves, centered rarefaction waves and contact discontinuity (also called vortex sheets).

Definition 3.3 (Shock waves) The triple (u_l, u_r, s) is called a p-shock if

$$\begin{array}{lll} (1) & \text{(Rankine -- Hugoniot condition)} & s(u_l - u_r) = f(u_l) - f(u_r), \\ (2) & \text{(Lax entropy condition)} & \lambda_p(u_r) < s < \lambda_p(u_l), \\ & & \lambda_{p-1}(u_l) < s < \lambda_{p+1}(u_r). \end{array}$$

Remark 1: Condition (2) implies that if we define the i-th characteristic curve by

$$\frac{dx_i(t)}{dt} = \lambda_i(u(x_i(t), t)),$$

then there are $(n+1)$ characteristic curves run into the shock and $(n-1)$ ones run away from it.

Example 4 $n = 1$. Suppose the shock is $x = st + x_0$, then the characteristic curves starting from x_1, x_2, which lies on the left and right hand side of the shock must run into the shock and no one leaves. See Figure 3.1.

$n = 2$. Consider the 1-shock $x = st + x_0$. Then the 1-characteristic curves starting from x_1, x_2 must run into the shock. Then it follows from the nonstrict hyperbolicity that the 2-characteristic curve starting from x_1 no way but run into the shock and then leaves it, and the 2-characteristic curve starting from x_2 must not run into the shock. See Figure 3.2.

Remark 2: Exactly as same as for $n = 1$, the Lax entropy conditions are the necessary and sufficient conditions for structural stability of the shock wave. That is, the jump continuity will persist under small perturbation, see A. Majda book for detail.

Definition 3.4 (Centered Rarefaction Waves) A function of the form $u = u(\frac{x}{t})$, which is Lipschitz continuous for $t > 0$, is called p-centered rarefaction wave if

(1) $\partial_t u + \partial_x f(u) = 0, \quad t > 0;$

(2) $\lambda_p \left(u\left(\frac{x}{t}\right)\right) = \frac{x}{t}, \quad \lambda_p(u_-) \leq \frac{x}{t} \leq \lambda_p(u_+).$

In other words,

$$u\left(\frac{x}{t}\right) = \begin{cases} u_-, & \frac{x}{t} \leq \lambda_p\left(u_-\right), \\ u(\frac{x}{t}), & \lambda_p\left(u_-\right) \leq \frac{x}{t} \leq \lambda_p\left(u_+\right), \\ u_+ & \frac{x}{t} \geq \lambda_p\left(u_+\right). \end{cases}$$

See Figure 3.3.

Remark: Clearly, if p-centered rarefaction wave exists, then $\lambda_p(u_-) \leq \lambda_p(u_+)$.

Definition 3.5 (Contact Discontinuity or Vortex sheets) A triple (u_-, u_+, s) is called p-contact discontinuity if

(1) $s(u_+ - u_-) = f(u_+) - f(u_-)$,

(2) $\lambda_p(u_-) = \lambda_p(u_+) = s$.

Remark 1: p-characteristic field has to be linearly degenerate to admit a contact discontinuity. See figure 3.4.

Remark 2: In view of computation, shock wave is easy to be observed since it has structural stability; while contact discontinuity is hard to be dealt with.

In the following, our basic assumptions are:

A. (3.1) is strictly hyperbolic;

B. Each characteristic field of (3.1) is either genuinely nonlinear or linearly degenerate.

As first step, our goal is to solve the Riemann problem for (3.1) with the following special initial data.

$$u(x, t = 0) = u^R(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{3.3}$$

Here $u_\pm$ are constant states.

Remark 1: Problem (3.1), (3.3) is called the Riemann problem just because Riemann originally studied the following problem in gas dynamics, which is also called shock tube problem.
Consider a long, thin, cylindrical tube containing a gas separated by a thin membrane. Let (u_l, ρ_l, p_l) and (u_r, ρ_r, p_r) denote the velocity, density and pressure on both sides of the membrane. Suppose at initial time, $u_l = u_r = 0, \quad \rho_l > \rho_r, \quad p_l > p_r$ are all constants (see Figure 3.5). The problem Riemann considered is to determine the motion of the gas after breaking the membrane at the initial time. (See Smoller's book)

Remark 2: The importance of the Riemann Problem is that the solutions to the Riemann Problem are scattering states both locally and globally for general solutions of (3.1).

To solve the Riemann problem (3.1), (3.3), we will use so-called wave curves to cover the state space $\Omega \subset R^n$. That is, given the left state u_-, we will look for all possible state u, which can be connected to u_- by either a shock wave, or a centered rarefaction wave, or a contact discontinuity.

Proposition 3.1 (Shock wave curve)
For fixed $u_0 \in R^n$, the R-H relations $s(u-u_0) = f(u) - f(u_0)$ define n-smooth curves $(u,s) = (u_k(\varepsilon), s_k(\varepsilon))$ for $|\varepsilon| \leq a_k, (k = 1, 2, \cdots, n), a_k > 0$, such that

(1) $u_k(0) = u_0, \quad s_k(\varepsilon = 0) = \lambda_k(u_0)$;

(2) $\dot{u}(0) = \frac{d}{d\varepsilon} u_k(\varepsilon)|_{\varepsilon=0} = r_k(u_0), \quad \ddot{u}(0) = \frac{d^2}{d\varepsilon^2} u_k(\varepsilon)|_{\varepsilon=0} = \dot{r}_k = \nabla r_k(u_0) \cdot r_k(u_0)$;

(3) k-family is genuinely nonlinear and we normalize it so that

$$\nabla \lambda_k(u_0) \cdot r_k(u_0) \equiv 1.$$

Then

$$\dot{s}_k(0) = \frac{d}{d\varepsilon} s_k \bigg|_{\varepsilon=0} = \frac{1}{2}$$

$$\lambda_k(u_k(\varepsilon)) < s_k(\varepsilon) < \lambda_k(u_0) \quad \text{iff} \quad \varepsilon < 0.$$

Proof

Step 1. Existence

Consider

$$s(u-u_0)=f(u)-f(u_0)=g(u,u_0)\,(u-u_0), \qquad (3.4)$$

where $g(u,u_0)=\int_0^1 f'(u_0+\theta\,(u-u_0))\,d\theta$. Clearly, $\lim_{u\to u_0} g(u,u_0)=\nabla f\,(u_0)\equiv A\,(u_0)$ and $g(u,u_0)$ is a smooth $n\times n$ matrix. By the assumption, $A\,(u_0)$ has n real distinct eigenvalues. Thus when u is close to $u_0, g(u,u_0)$ must have n real distinct eigenvalues $\bar{\lambda}_k(u,u_0)$ with corresponding right (left) eigenvector $\bar{r}_k(u)\,(\bar{l}_k\,(u))$.

Then (3.4) is equivalent to

$$(g(u,u_0) - sI)(u - u_0) = 0.$$

So R-H condition is satisfied if and only if there exists $k, k = 1, 2, \cdots, n$, such that $s = \bar{\lambda}_k(u)$ and $u - u_0 \parallel \bar{r}_k(u)$, which implies

$$\bar{l}_i(u) \cdot (u - u_0) = 0, \quad i \neq k.$$

That is, u must satisfies

$$\Phi(u) \equiv \tilde{L}(u) \cdot (u - u_0) = 0,$$

where

$$\tilde{L}(u) = (\bar{l}_1(u), \cdots, \bar{l}_{k-1}(u), \bar{l}_{k+1}(u), \cdots, \bar{l}_n(u))^t.$$

Clearly, $\Phi(u_0) = 0, \quad d\,\Phi(u_0) = \tilde{L}(u_0)$ has rank $n-1$. So by implicit function theorem, there exists a real number ε such that $u = u_k(\varepsilon)$ defined in a small neighborhood $|\varepsilon| < a_k (0 < a_k \ll 1)$ such that

$$u_k(0) = u_0, \quad \Phi(u_k(\varepsilon)) \equiv 0$$

and

$$u_k(\varepsilon) - u(0) \parallel \bar{r}_k(u).$$

We define $s = s_k(\varepsilon) = \bar{\lambda}_k(u_k(\varepsilon))$.

Step 2. Properties of the shock locus

By step 1, we have

$$s_k(\varepsilon)(u_k(\varepsilon) - u_0) = (f(u_k(\varepsilon)) - f(u_0)). \tag{3.5}$$

By definition of the right eigenvalue, we also have

$$\nabla f(u_k(\varepsilon))\, r_k(u_k(\varepsilon)) = \lambda_k\,(u_k(\varepsilon))\, r_k(u_k(\varepsilon)). \tag{3.6}$$

From (3.5), one has

$$\dot{s}_k\,(u_k - u_0) + s_k\,\dot{u}_k = f^{'}\,(u_k)\,\dot{u}_k, \tag{3.7}$$

$$\begin{aligned} &\ddot{s}_k\,(u_k - u_0) + 2\,\dot{s}_k\,\dot{u}_k + s_k\,\ddot{u}_k \\ &= \nabla^2 f(u_k)\,(\dot{u}_k, \dot{u}_k) + f^{'}\,(u_k)\,\ddot{u}_k \end{aligned} \tag{3.8}$$

From (ref3.6), one has

$$\nabla^2 f(u_k)(\dot{u}_k, r_k) + f^{'}(u_k)\,\dot{r}_k = (\nabla\lambda_k(u_k)\dot{u}_k)\,r_k(u_k) + \lambda_k(u_k)\,\dot{r}_k. \tag{3.9}$$

Here we omit the parameter ε for simplicity. Recall that if $f = (f_1, f_2, \cdots, f_n), f_i = f_i(u)$, and $H(f_i)$ denotes the Hessian matrix of f_i, then $\nabla^2 f\,(r_i, r_i)$ is the column vector defined by

$$\nabla^2 f(r_i, r_i) = \begin{pmatrix} r_i^t\, H(f_1)\, r_i \\ r_i^t\, H(f_2)\, r_i \\ \vdots \\ r_i^t\, H(f_n)\, r_i \end{pmatrix}.$$

Notation $f^{'}$ means the gradient of f, also denoted by ∇f.
Set $\varepsilon = 0$ in (3.6) and (3.7), note that $u_k(0) = u_0$, it yields

$$(f^{'}(u_0) - \lambda_k(u_0)I)\, r_k(u_0) = 0,$$

and

$$(f^{'}(u_0) - s_k(0)I)\, \dot{u}_k(0) = 0.$$

Therefore, after normalizing, we get

$$s_k(0) = \lambda_k(u_0), \quad \dot{u}_k(0) = r_k(u_0). \tag{3.10}$$

Then, set $\varepsilon = 0$ in (3.8) and use (3.10) to give

$$2\,\dot{s}_k(0) r_k(u_0) + \lambda_k(u_0)\, \ddot{u}_k(0) = \nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)) + f^{'}(u_0)\, \ddot{u}_k(0). \tag{3.11}$$

Applying $l_k(u_0)$ on both hand side of above equation, one has

$$\begin{aligned} & 2\,\dot{s}_k(0)\,l_k(u_0)\,r_k(u_0) + \lambda_k(u_0)\,l_k(u_0)\,\ddot{u}_k(0) \\ = & \, l_k(u_0)\,\nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)) + \lambda_k(u_0)\,l_k(u_0)\,\ddot{u}_k(0). \end{aligned}$$

That is,

$$2\,\dot{s}_k(0) = l_k(u_0)\,\nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)). \tag{3.12}$$

Noting that

$$\nabla f(u_k)\,r_k(u_k) = \lambda_k\,(u_k)\,r_k\,(u_k),$$

one has

$$\begin{aligned} & \nabla^2 f(u_k)\,(\dot{u}_k, r_k(u_k)) + \nabla f(u_k)\,\dot{r}_k(u_k) \\ = & \,\nabla\,\lambda_k\,(u_k)\cdot\dot{u}_k\,r_k(u_k) + \lambda_k\,(u_k)\,\dot{r}_k\,(u_k). \end{aligned}$$

So

$$l_k(u_0) \cdot \nabla^2 f(u_0)(r_k(u_0), r_k(u_0)) = l_k(u_0) \cdot \nabla \lambda_k(u_0) \cdot r_k(u_0)\, r_k(u_0)$$

and

$$l_k(u_0) \nabla^2 f(u_0)(r_k(u_0), r_k(u_0)) = \nabla \lambda_k(u_0) \cdot r_k(u_0).$$

By our assumptions, k-family is genuinely nonlinear, and

$$\nabla \lambda_k(u_0) \cdot r_k(u_0) = 1$$

Therefore, it deduces from (3.12) that

$$\begin{aligned} 2\,\dot{s}_k(0) &= \nabla \lambda_k(u_0) \cdot r_k(u_0) = 1, \\ \dot{s}_k(0) &= \tfrac{1}{2}. \end{aligned}$$

Then, the equation (3.11) becomes

$$r_k(u_0) + \lambda_k(u_0)\, \ddot{u}_k(0) = \nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)) + f^{'}\,(u_0)\, \ddot{u}_k(0). \tag{3.13}$$

On the other hand, from (3.6), we have

$$\begin{aligned} &\nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)) + f^{'}\,(u_0)\, \dot{r}_k(u_0) \\ &= (\nabla \lambda_k(u_0) \cdot r_k(u_0))\, r_k(u_0) + \lambda_k(u_0) \cdot \dot{r}_k(u_0). \end{aligned}$$

which is

$$\nabla^2 f(u_0)\,(r_k(u_0), r_k(u_0)) + f^{'}\,(u_0)\, \dot{r}_k(u_0) = r_k(u_0) + \lambda_k(u_0) \cdot \dot{r}_k(u_0). \tag{3.14}$$

From (3.13), (3.14), it reduces

$$\nabla f(u_0)\,(\ddot{u}_k - \dot{r}_k) = \lambda_k(u_0)\,(\ddot{u}_k - \dot{r}_k).$$

Therefore,

$$\ddot{u}_k - \dot{r}_k \parallel r_k(u_0),$$

$$\ddot{u}_k = \dot{r}_k + c\, r_k(u_0),$$

where c is a constant. After reparameterizing the curve again, we get

$$\ddot{u}_k = \dot{r}_k.$$

Until now, we have gotten n-smooth curves $(u_k(\varepsilon), s_k(\varepsilon))$ for $|\varepsilon| < a_k$ satisfying properties (1), (2) of the proposition, and

$$\dot{s}_k(0) = \frac{1}{2}.$$

In the following, we will prove the entropy conditions as stated in (3) of the proposition.

Step 3. Entropy condition

Set $\Phi(\varepsilon) = s_k(\varepsilon) - \lambda_k(u_0)$.
Then $\Phi(0) = 0$.

$$\dot{\Phi}(\varepsilon)|_{\varepsilon=0} = \dot{s}_k(\varepsilon)|_{\varepsilon=0} = \frac{1}{2}$$

Consequently, one has

$$\Phi(\varepsilon) < 0$$

if and only if $\varepsilon < 0$.
Now set $\psi(\varepsilon) = \lambda_k(u_k(\varepsilon)) - s_k(\varepsilon)$. Then, clearly,

$$\begin{aligned} \psi(0) &= 0 \\ \dot{\psi}(\varepsilon)|_{\varepsilon=0} &= \nabla \lambda_k(u_k(\varepsilon))\, \dot{u}_k|_{\varepsilon=0} - \dot{s}_k|_{\varepsilon=0} = 1 - \tfrac{1}{2} = \tfrac{1}{2}. \end{aligned}$$

So $\psi(\varepsilon) < 0$ if and only if $\varepsilon < 0$.

Thus, we have obtained

$$\lambda_k(u_k(\varepsilon)) < s_k(\varepsilon) < \lambda_k(u_0)$$

if and only if $\varepsilon < 0$, as required by our proposition.

So far, for fixed $u_0 \epsilon R^n$, we have constructed n-smooth curves $(u_k(\varepsilon), s_k(\varepsilon))$ connecting u_0 in the neighborhood of u_0, and satisfying entropy conditions for $\varepsilon < 0$. This is called shock curve connecting u_0. The following is about rarefaction wave curve connecting u_0 in the neighborhood of u_0.

Define $u_k^R(\varepsilon)$ to be the vector field associated with $r_k(u)$, i.e.

$$\begin{cases} \frac{d}{d\varepsilon} u_k^R(\varepsilon) = r_k \left(u_k^R(\varepsilon) \right), \\ u_k^R(\varepsilon = 0) = u_0. \end{cases}$$

The local existence of $u_k^R(\varepsilon)$ on ε is clear. The we have

Proposition 3.2 (Rarefaction Wave Curve)

(1) If k-characteristic field is genuinely nonlinear, define

$$u_k^R\left(\frac{x}{t}\right) = \begin{cases} u_0, & \frac{x}{t} \leq \lambda_k(u_0), \\ u_k^R\left(\frac{x}{t} - \lambda_k(u_0)\right), & \lambda_k(u_0) \leq \frac{x}{t} \leq \lambda_k(u_0) + \tilde{\varepsilon}, \\ u_k^R(\tilde{\varepsilon}), & \frac{x}{t} \geq \lambda_k(u_0) + \tilde{\varepsilon}. \end{cases}$$

where $0 < \tilde{\varepsilon} \ll 1$. Then u_k^R is the k - centered rarefaction wave connecting u_0 to $u_k^R(\tilde{\varepsilon})$;

(2) If k-characteristic field is linearly degenerate, define

$$u^R\left(\frac{x}{t}\right) = \begin{cases} u_0, & \frac{x}{t} < \lambda_k(u_0), \\ u_k^R(\varepsilon), & \frac{x}{t} > \lambda_k(u_0). \end{cases}$$

Then $u^R\left(\frac{x}{t}\right)$ gives the k-contact discontinuity connecting u_0 to $u_k^R(\varepsilon)$.

Proof

(1) Let $\varepsilon = \frac{x}{t} - \lambda_k(u_0)$. Then we have

$$\frac{d}{d\varepsilon}\lambda_k\left(u_k^R(\varepsilon)\right) = \nabla\lambda_k\left(u_k^R(\varepsilon)\right)\cdot\frac{d}{d\varepsilon}u_k^R(\varepsilon)$$
$$= \nabla\lambda_k\left(u_k^R(\varepsilon)\right)\cdot r_k\left(u_k^R(\varepsilon)\right) \equiv 1$$

so

$$\lambda_k\left(u_k^R(\varepsilon)\right) = \lambda_k\left(u_k^R(0)\right) + \varepsilon = \lambda_k(u_0) + \frac{x}{t} - \lambda_k(u_0)$$
$$= \frac{x}{t}.$$

Denote $u(x,t) = u_k^R\left(\frac{x}{t} - \lambda_k(u_0)\right)$. Then

$$\begin{aligned}
\partial_t u + \partial_x f(u) &= \frac{d}{d\varepsilon} u_k^R \left(-\frac{x}{t^2}\right) + f'\left(u_k^R\right) \cdot \frac{d}{d\varepsilon} u_k^R \cdot \frac{1}{t} \\
&= -\frac{x}{t^2} r_k\left(u_k^R(\varepsilon)\right) + \frac{1}{t} f'\left(u_k^R\right) \cdot r_k\left(u_k^R(\varepsilon)\right) \\
&= -\frac{x}{t^2} r_k\left(u_k^R(\varepsilon)\right) + \frac{1}{t} \lambda_k\left(u_k^R(\varepsilon)\right) r_k\left(u_k^R(\varepsilon)\right) \\
&= 0.
\end{aligned}$$

Therefore u_k^R is the k-centered rarefaction wave.

(2) By definition, we need to prove

$$\lambda_k(u_0) = \lambda_k\left(u_k^R(\varepsilon)\right) \tag{3.15}$$

and

$$\lambda_k(u_0)\left(u_k^R(\varepsilon) - u_0\right) = f\left(u_k^R(\varepsilon)\right) - f(u_0). \tag{3.16}$$

Since k-characteristic field is linearly degenerate, one has

$$\begin{aligned} \frac{d}{d\varepsilon}\left(\lambda_k(u_0) - \lambda_k(u_k^R(\varepsilon))\right) &= -\nabla\lambda_k\left(u_k^R(\varepsilon)\right)\frac{d}{d\varepsilon}u_k^R(\varepsilon) \\ &= -\nabla\lambda_k\left(u_k^R(\varepsilon)\right)\cdot r_k\left(u_k^R(\varepsilon)\right) \equiv 0. \end{aligned}$$

Therefore,

$$\lambda_k(u_0) - \lambda_k\left(u_k^R(\varepsilon)\right) = \lambda_k(u_0) - \lambda_k\left(u_k^R(\varepsilon = 0)\right) \equiv 0. \quad (3.17)$$

This is (3.15).

Set $\Phi(\varepsilon) = \lambda_k(u_0)\left(u_k^R(\varepsilon) - u_0\right) - \left(f(u_k^R(\varepsilon)) - f(u_0)\right)$.
Then

$$\begin{aligned} \frac{d}{d\varepsilon}\Phi(\varepsilon) &= \lambda_k(u_0)\,\frac{d}{d\varepsilon}\,u_k^R(\varepsilon) - \nabla f\left(u_k^R(\varepsilon)\right)\frac{d}{d\varepsilon}\,u_k^R(\varepsilon) \\ &= \left(\lambda_k(u_0) - \lambda_k(u_k^R(\varepsilon))\right)\, r_k\left(u_k^R(\varepsilon)\right) \\ &= 0 \qquad\qquad \text{by } (3.17). \end{aligned}$$

Noticing that

$$\Phi(0) = 0.$$

we obtain $\Phi(\varepsilon) \equiv 0$, which is (3.16).

Now for fixed $u_0 \,\epsilon\, \Omega$, we can find a neighborhood N of u_0 in Ω so that there is a shock wave curve $u_k^S(\varepsilon)$ through u_0 in N satisfying the Lax entropy condition on $\varepsilon < 0$, and a rarefaction wave curve $u_k^R(\varepsilon)$ going through u_0 in N, provided that each characteristic field is either genuinely nonlinear or linearly degenerate. We define a k-wave curve by combining one sided branches of wave curves.

Definition 3.6 (Wave curve) A k-wave curve through u_0 is a $C^{2,1}$ curve $T^k(\varepsilon)u_0$ defined to be

(1) If k-field is genuinely nonlinear,

$$u = T^k(\varepsilon)u_0 = T_k(\varepsilon, u_0) = \begin{cases} u_k^S(\varepsilon), & \varepsilon \leq 0 \\ u_k^R(\varepsilon), & \varepsilon > 0 \end{cases}$$

(2) If k-field is linearly degenerate, $u = T^k(\varepsilon)u_0 = u_k^C(\varepsilon)$, where u_k^C denotes the k-contact discontinuity wave.

We show that we can connect two nearby states by combination of k-wave curves. The theorem is stated as follows.

Theorem 3.1 (Lax) Let the system is strictly hyperbolic, and each field is either genuinely nonlinear or linearly degenerate on a region $\Omega \subset R^n$. Assume $u_- \epsilon \Omega$. Then there is a small neighborhood N of $u_- \epsilon \Omega$ such that for any $u_+ \epsilon N$, the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t = 0) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0 \end{cases} \end{cases}$$

has a solution. Further, this solution consists of at most $(n + 1)$ constant states separated by shock, centered rarefaction wave and contact discontinuity. There is precisely one such solution.

The proof of this theorem follows simply from inverse function theorem.

Proof: By Proposition 3.1 and 3.2, there exists a neighborhood N and $a > 0$ such that $T^k_{\varepsilon_k} : N \to R^n$ for $|\varepsilon_k| < a, \quad k = 1, 2, \cdots, n$, are well defined and $C^{2,1}$ with the property that for any $u \,\epsilon\, N$, u can be joint to $T^k_{\varepsilon_k}\, u$ on the right by either a k-shock or a k-centered rarefaction wave or k-contact discontinuity.

Now let $u_l \,\epsilon\, N$ be fixed. Define
$\mho = \{\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in R^n : |\varepsilon_k| < a, 1 \le k \le n\}$. Let
$T : \mho \to R^n$ be defined as

$$T(\varepsilon) = T^n_{\varepsilon_n}(T^{n-1}_{\varepsilon_{n-1}}(\cdots(T^2_{\varepsilon_2}(T^1_{\varepsilon_1}(u_l)))\cdots)) = T^n_{\varepsilon_n} \circ T^{n-1}_{\varepsilon_{n-1}} \circ \cdots \circ T^1_{\varepsilon_1}\, u_l$$

Our goal is to show that for any $u_r \epsilon \Omega$ sufficiently close to u_l, $|u_r - u_l| < \delta$, there is $\bar{\varepsilon} = \bar{\varepsilon}(\delta) \epsilon \mho$ such that $T(\bar{\varepsilon})\, u_l = u_r$. To see this, define $F(\varepsilon) = T(\varepsilon)\, u_l - u_l$. Since $F(0) = 0$ and rank $dF(0) = \text{rank}\,(r_1(u_l), r_2(u_l), \cdots, r_n(u_l)) = n$, by inverse function theorem, there is $\delta > 0$ such that, for any $u_r \epsilon \Omega$ with $|u_r - u_l| < \delta$, there exists $\varepsilon \epsilon \mho$ such that $F(\varepsilon) = u_r - u_l$, that is, $T^n_{\varepsilon_n} \circ \cdots \circ T^1_{\varepsilon_1}(u_l) = u_r$. So the theorem follows.

Remark:

1. We may not solve the Riemann problem in two general constant states. However, for gas dynamics, the Riemann problem can be solved globally. For details see the book of Joel Smoller, Shock Waves and Reaction - Diffusion Equation, Springer - Verlag, Chapter 18.
2. Similar results can be obtained for system without assuming that the field is genuinely nonlinear. For instance, see Liu, Tai Ping, Admissible solutions of hyperbolic conservation laws, Memoirs of the American Mathematical Society, 30 (1981), no. 240 iv +78pp.

§3.2 Estimates on Wave Interactions

In scalar conservation laws, for any initial data consisting of three constant states (u_l, u_m, u_r), we have discussed all possible wave interaction in Chapter 1. It becomes a shock for interaction of two shocks, a rarefaction wave for those of two rarefaction waves, a weak shock if the shock is stronger than the rarefaction wave, and a weak rarefaction wave if the shock is weaker than the rarefaction wave. For systems of conservation laws, one should imagine that there are difficulties for wave interaction. Fortunately, because any two waves do not interact each other again after they have interacted, the Riemann solution should determine the long time asymptotics of a general solution just as in the scalar case.

Lemma 3.1 Let (u_-, u_+) be solved with $\mu = (\mu_1, \cdots, \mu_n)$, i.e.,

$$u_+ = T_\mu u_- = T^n_{\mu_n} \circ \cdots \circ T^1_{\mu_1} u_-,$$

then

$$u_+ = u_- + \sum_{i=1}^{n} \mu_i r_i + \frac{1}{2} \sum_{i=1}^{n} \mu_i^2 \nabla r_i \cdot r_i + \sum_{1 \leq i < j \leq n} \mu_i \mu_j \nabla r_j \cdot r_i + o(|\mu|^3) \tag{3.18}$$

here all $r_i, \nabla r_j \cdot r_i$ are evaluated at u_-.

Proof: Set $u_i = T^i_{\mu_i} u_{i-1}, \quad i = 1, 2, \cdots, n, \quad u_0 = u_-, \quad u_n = u_+$.
From Proposition 3.1 & 3.2,

$$\begin{aligned} u_i &= T^i_{\mu_i} u_{i-1} \\ &= u_{i-1} + \mu_i r_i(u_{i-1}) + \frac{1}{2}\mu_i^2 \nabla r_i \cdot r_i(u_{i-1}) + o(|\mu|^3) \\ &= u_{i-1} + \mu_i r_i(u_-) + \mu_i(r_i(u_{i-1}) - r_i(u_-)) \\ &\quad + \frac{1}{2}\mu_i^2 \nabla r_i \cdot r_i(u_-) + o(|\mu|^3) \end{aligned}$$

since

$$
\begin{aligned}
r_i(u_{i-1}) - r_i(u_-) &= \sum_{j=1}^{i-1} r_i(u_j) - r_i(u_{j-1}) \\
&= \sum_{j=1}^{i-1} \nabla r_i \cdot r_j(u_{j-1})\, \mu_j + o(|\mu|^2) \\
&= \sum_{j=1}^{i-1} \mu_j\, \nabla r_i \cdot r_j(u_-) + o(|\mu|^2)
\end{aligned}
$$

hence

$$
u_i = u_{i-1} + \mu_i\, r_i(u_-) + \frac{1}{2}\mu_i^2\, \nabla r_i \cdot r_i(u_-) + \sum_{j=1}^{i-1} \mu_i\, \mu_j\, \nabla r_i \cdot r_j(u_-) + o(|\mu|^3) \tag{3.19}
$$

By induction of (3.19) we get

$$u_k = u_- + \sum_{i \le k} \mu_i \, r_i(u_-) + \sum_{i<j\le k} \mu_i \, \mu_j \, \nabla r_j \cdot r_i + \frac{1}{2} \sum_{i \le k} \mu_i^2 \, \nabla r_i \cdot r_i + o(|\mu|^3)$$

for $k = 1, 2, \cdots, n$. This gives the lemma.

Lemma 3.2 (Rough Estimate of Wave Interaction)
For any fixed $u_l \,\epsilon\, \Omega$, the result of interaction of two adjacent Riemann solution $\alpha((u_l, u_m))$, $\beta((u_m, u_r))$ is a simple Riemann solution $\varepsilon((u_l, u_r))$. Then $\varepsilon = \varepsilon(\alpha, \beta)$ is $C^{2,1}$, that is, each second partial derivatives are Lipschitz continuous, and satisfies

$$\sum_{i=1}^{n} \varepsilon_i \, r_i = \sum_{i=1}^{n} (\alpha_i+\beta_i) r_i + \sum_{j \geq k} \alpha_j \, \beta_k \, (\nabla r_k \cdot r_j - \nabla r_j \cdot r_k) + o(|\alpha| + |\beta|)^3 \tag{3.20}$$

In particular, $\varepsilon_i = \alpha_i + \beta_i + O\left(|\alpha| \cdot |\beta| + (|\alpha| + |\beta|)^3\right)$. If we define $R_i = r_i \cdot \nabla$, then (3.20) can be written as

$$\sum_{i=1}^{n} \varepsilon_i \, R_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) R_i + \sum_{j \geq k} \alpha_j \, \beta_k \, [R_j, R_k] + O(1)(|\alpha| + |\beta|)^3$$

where $[R_j, R_k] = R_j \, R_k - R_k \, R_j$ denotes the Lie bracket of two vector fields.

Proof: By Lemma 3.1,

$$u_m = u_l + \sum_{i=1}^{n} \alpha_i \, r_i + \frac{1}{2} \sum \alpha_i^2 \, \nabla r_i \cdot r_i + \sum_{i<j} \alpha_i \, \alpha_j \, \nabla r_j \cdot r_i + O(|\alpha|^3) \qquad (3.21)$$

$$\begin{aligned} u_r &= u_m + \sum_{i=1}^{n} \beta_i \, r_i(u_m) + \frac{1}{2} \sum \beta_i^2 \, \nabla r_i(u_m) \cdot r_i(u_m) \\ &\quad + \sum_{i<j} \beta_i \, \beta_j \, \nabla r_j(u_m) \cdot r_i(u_m) + O(|\alpha|^3) \qquad (3.22) \end{aligned}$$

where r_i, ∇r_i are evaluated at u_l for convenience.

Substitute (3.21) into (3.22), by the fact $r_i(u_m) = r_i(u_l) + \sum_{j=1}^{n} \alpha_j \nabla r_i \cdot r_j + O(|\alpha|^2)$, we have

$$\begin{aligned} u_r &= u_l + \sum_{i=1}^{n} \alpha_i r_i + \frac{1}{2}\sum_{i=1}^{n} \alpha_i^2 \nabla r_i \cdot r_i + \sum_{i<j} \alpha_i \alpha_j \nabla r_j \cdot r_i + \sum_{i=1}^{n} \beta_i r_i(u_m) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} \beta_i^2 \nabla r_i \cdot r_i + \sum_{i<j} \beta_i \beta_j \nabla r_j \cdot r_i + O(|\alpha| + |\beta|)^3 \\ &= u_l + \sum_{i=1}^{n} (\alpha_i + \beta_i) r_i + \frac{1}{2}\sum_{i=1}^{n} (\alpha_i + \beta_i)^2 \nabla r_i \cdot r_i + \sum_{i<j} (\alpha_i \alpha_j + \beta_i \beta_j) \nabla r_j \cdot r_i \\ &\quad + \sum_{i \neq j} \alpha_i \beta_j \nabla r_j \cdot r_i + O(|\alpha| + |\beta|)^3 \end{aligned} \tag{3.23}$$

Since $\varepsilon = \varepsilon(\alpha, \beta)$ is a smooth function $C^{2,1}$, by the wave curve definition, $\varepsilon(0,0)$, $|\varepsilon| = O(1)(|\alpha| + |\beta|)$ and $|\varepsilon|^n = O(1)(|\alpha| + |\beta|)^n$. On the other hand, ε solves (u_l, u_r), hence

$$u_r = u_l + \sum_{i=1}^{n} \varepsilon_i r_i + \frac{1}{2}\sum \varepsilon_i^2 \nabla r_i \cdot r_i + \sum_{i<j} \varepsilon_i \varepsilon_j \nabla r_j \cdot r_i + O(|\varepsilon|^3) \tag{3.24}$$

Compare (3.23), (3.24) with $\varepsilon_i = \alpha_i + \beta_i + O(1)(|\alpha| + |\beta|)^2$, we obtain

$$\begin{aligned}
\sum \varepsilon_i\, r_i &= \sum(\alpha_i + \beta_i)\, r_i + \sum_{i<j} (\alpha_i\, \alpha_j + \beta_i\, \beta_j)\, \nabla r_j \cdot r_i + \sum_{i<j} \alpha_i\, \beta_j\, \nabla r_j \cdot r_i + \sum_{i>j} \alpha_i\, \beta_j\, \nabla r_j \cdot r_i \\
&\quad - \sum_{i<j} (\alpha_i + \beta_i)(\alpha_j + \beta_j)\, \nabla r_j \cdot r_i + O(|\alpha| + |\beta|)^3 \\
&= \sum(\alpha_i + \beta_i)\, r_i + \sum_{i>j} \alpha_i\, \beta_j\, \nabla r_j \cdot r_i - \sum_{i<j} \alpha_j\, \beta_i\, \nabla r_j \cdot r_i + O(|\alpha| + |\beta|)^3 \\
&= \sum(\alpha_i + \beta_i)\, r_i + \sum_{i<j} \alpha_j\, \beta_i\, (\nabla r_i \cdot r_j - \nabla r_j \cdot r_i) + O(|\alpha| + |\beta|)^3
\end{aligned}$$

This shows (3.20).

Remark: This is not the optimal estimate. For example, $\alpha = (\alpha_1, 0)$, $\beta = (0, \beta_2)$ with $\alpha_1 < 0$, $\beta_2 < 0$, then $\varepsilon = (\alpha_1, \beta_2)$ and $\varepsilon_i = \alpha_i + \beta_i$, $i = 1, 2$. This example illustrates that we may get better estimate on the third order term. From the same idea of the second order term, we need only to compute those waves which will produce interaction.

Definition 3.7 (Approaching Waves) Elementary waves α_j and β_k are said to be approaching if

(1) if $j \neq k$, then $j > k$.

(2) if $j = k$, then one of them must be a shock wave, i.e., either $\alpha_j < 0$ or $\beta_k < 0$.

Lemma 3.3 Under the same conditions as in Lemma 3.2, then

$$\sum \varepsilon_i R_i = \sum (\alpha_i + \beta_i) R_i + \sum_{j>k} \alpha_j \beta_k [R_j, R_k] + D(\alpha, \beta) O(S(\alpha, \beta)) \tag{3.25}$$

where $S(\alpha, \beta) = max \{|\alpha_i|, |\beta_i|\}$, $D(\alpha, \beta) = \sum' |\alpha_j| \cdot |\beta_k|$, the summation $\sum'$ is taken over all approaching waves.

Proof: Define $F : R^{2n} \to R^n$ by

$$F(\alpha, \beta) = \sum \varepsilon_i(\alpha, \beta) R_i - \left(\sum (\alpha_i + \beta_i) R_i + \sum_{j>k} \alpha_j \beta_k [R_j, R_k] \right).$$

We claim that $|F(\alpha, \beta)| \leq C\, D(\alpha, \beta) \cdot S(\alpha, \beta)$. It can be realized by the following two steps.

Step 1: If $D(\alpha,\beta)=0$, then $F(\alpha,\beta)=0$.
To see this, if either $\alpha\equiv 0$ or $\beta\equiv 0$, then clearly $F(\alpha,\beta)=0$. If $\alpha_i\neq 0$ for some i, then by $D(\alpha,\beta)=0$, $\beta_j=0$ for all $j<i$, and either $\alpha_i\cdot\beta_i=0$ or $\alpha_i>0$, $\beta_i>0$. If i is chosen to be maximum number so that $\alpha_i\neq 0$, for the case $\alpha_i>0$, $\beta_i>0$, then the interaction wave ε_i is simply combining the rarefaction waves into one. Hence $\varepsilon_i=\alpha_i+\beta_i$ and $F(\alpha,\beta)=0$. It finishes step 1.

Step 2: By definition, $F\,\epsilon\,C^{2,1}$. So by Lemma 3.2, $F(0,0)=F(\alpha,0)=F(0,\beta)=0, F'_\alpha(0,0)=0$, $F'_\beta(0,0)=0$, $F''(0,0)=0$. It follows that $F(\alpha,\beta)=\sum\alpha_i\,\beta_j\,\Phi_{ij}\,(\alpha,\beta)$, here $\Phi_{ij}\,(\alpha,\beta)$ is Lipschitz continuous function. In fact,

$$
\begin{aligned}
F(\alpha,\beta) &= \sum_{i,j}\left[F(\alpha_1,\cdots,\alpha_i,0,\cdots,0,\beta_j,\cdots,\beta_n) - F(\alpha_1,\cdots,\alpha_i,0,\cdots,0,\beta_{j+1},\cdots,\beta_n)\right.\\
&\quad \left. -F(\alpha_1,\cdots,\alpha_{i-1},0,\cdots,0,\beta_j,\cdots,\beta_n) + F(\alpha_1,\cdots,\alpha_{i-1},0,\cdots,0,\beta_{j+1},\cdots,\beta_n)\right]\\
&= \sum_{i,j}\Big[\beta_j \int_0^1 F'_{\beta_j}(\alpha_1,\cdots,\alpha_i,0,\cdots,0,t\,\beta_j,\beta_{j+1},\cdots,\beta_n)\,dt\\
&\quad -\beta_j \int_0^1 F'_{\beta_j}(\alpha_1,\cdots,\alpha_{i-1},0,\cdots,0,t\,\beta_j,\beta_{j+1},\cdots,\beta_n)\,dt\Big]\\
&= \sum_{i,j} \alpha_i\,\beta_j \int_0^1\int_0^1 F''_{\alpha_i\,\beta_j}(\alpha_1,\cdots,\alpha_{i-1},s\,\alpha_i,0,\cdots,0,t\,\beta_j,\beta_{j+1},\cdots,\beta_n)\,ds\,dt
\end{aligned}
$$

and

$$
\Phi_{ij}(\alpha,\beta) = \int_0^1\int_0^1 F''_{\alpha_i\,\beta_j}(\alpha_1,\cdots,\alpha_{i-1},s\,\alpha_i,0,\cdots,0,t\,\beta_j,\beta_{j+1},\cdots,\beta_n)\,ds\,dt.
$$

Φ_{ij} satisfies $\Phi_{ij}(0,0)=0$, $|\Phi_{ij}(\alpha,\beta)| \le O(1)(|\alpha|+|\beta|)$.

Note that if α_i, β_j are not approaching, i.e., $i < j$ and either $\alpha_i \cdot \beta_i = 0$ or $\alpha_i > 0$, $\beta_i > 0$, then $\Phi_{ij}(\alpha, \beta) = 0$ since

$$\begin{array}{ll} & F(\alpha_1, \cdots, \alpha_i, 0, \cdots, 0, \beta_j, \cdots, \beta_n) \quad = F(\alpha_1, \cdots, \alpha_i, 0, \cdots, 0, \beta_{j+1}, \cdots, \beta_n) \\ = & F(\alpha_1, \cdots, \alpha_{i-1}, 0, \cdots, 0, \beta_j, \cdots, \beta_n) \quad = F(\alpha_1, \cdots, \alpha_{i-1}, 0, \cdots, 0, \beta_{j+1}, \cdots, \beta_n) \\ = & 0 \end{array}$$

Therefore

$$\begin{aligned} |F(\alpha, \beta)| &= \left| \sum_{Approaching} \alpha_i \, \beta_j \, \Phi_{ij}(\alpha, \beta) \right| \\ &\leq O(1) \, D(\alpha, \beta) \cdot (|\alpha|, |\beta|). \end{aligned}$$

§3.3 Glimm Scheme and its Stability

In this section we give a description of the Glimm scheme to solve the following general Cauchy problem

$$\partial_t u + \partial_x f(u) = 0, \tag{3.26}$$

$$u(x, t = 0) = u_0(x) \tag{3.27}$$

We suppose that (3.26) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

Before Glimm, people only worked on special initial data for special systems. But for very general initial data, the break through is really due to J. Glimm (1966).

We have known that for Riemann data

$$u(x, t = 0) = u^R(x) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0, \end{cases}$$

the Riemann problem has a unique solution which is the superposition of constant states separated by k-elementary waves, $k = 1, 2, \cdots, n$, as long as $|u_+ - u_-| \ll 1$.

In the space of functions of bounded total variation, Glimm uses the Riemann solution as the building blocks of general solution. The essential idea is his realization of wave interactions. The success of Glimm scheme is mainly due to two elements: 1) Glimm functional; 2) idea of random choice.

(1) Random choice method
To make thing easy going, we introduce the method step by step.

1. Let U_1 be a neighborhood of 0. First choose a neighborhood U_3 (bounded open set), such that for any $u_l, u_r \in U_3 \subset U_2$, the Riemann problem (u_l, u_r) has a solution with intermediate states $u_1, u_2, \cdots, u_{n-1} \in U_2$ with $\bar{U}_2 \subset U_1$. (See Figure 3.6)

   Here we do not have maximum principle, so the Riemann solution generally lies in a slightly bigger set than U_3. Only for special systems, (u_l, u_r) is in the same region as U_3.

2. Now choose positive constants C so large that CFL (Courant - Friedrichs - lewy) condition holds

$$\Lambda = \sup\{|\lambda_2(u)|,\ u \,\epsilon\, U_2,\ 1 \leq i \leq n\} < C = \frac{\Delta x}{\Delta t}, \qquad (3.28)$$

   where $\Delta x, \Delta t$ are the space step and time step, respectively. In the construction of the sequence of approximate solutions, we will let Δx tend to zero.

3. Let a sequence $\theta = \{\theta_i\}_{i=1}^{\infty}$ be a equally distributed sequence of random numbers in (-1,1).
   A sequence is equally distributed means that given any length, the probability that a number is to be in any interval of this length is the same, just like the Brownian motion.

4. For convenience of description, we give some notations. The lattice is defined to be

$$Y^+ = \{(m,n)\,\epsilon\, Z \times Z, m+n = 0 \pmod 2, n \geq 0\}.$$

The mesh points are chosen to be

$$\begin{aligned} a_m^n \in \Phi &= \textstyle\prod_{(m,n)\in Y^+}[(m-1)\Delta x, (m+1)\Delta x] \times \{n\,\Delta t\}, \\ a_m^n &= ((m+\theta_n)\,\Delta x,\; n\,\Delta t). \end{aligned}$$

(See Figure 3.7)

5 Approximate solution

This is constructed by induction on $n \in Z^+$ for each strip $R^1 \times [n\,\Delta t, (n+1)\,\Delta t]$. Inductively, if we have already defined $u(x,t)$, $t \leq (n-1)\,\Delta t$, then one can define $u(x,t)$ on $t < n\,\Delta t$ as follows:

$$\text{for } n + m = 0 \,(\text{mod}\, 2), \text{ set}$$

$$v(x,(n-1)\Delta t) = \begin{cases} u\left(a_{m-1}^{n-1}\right), & (m-1)\Delta x \leq x \leq m\,\Delta x, \\ u\left(a_{m+1}^{n-1}\right), & m\,\Delta x \leq x \leq (m+1)\Delta x, \end{cases}$$

then let $u(x,t)$, $(n-1)\,\Delta t \leq t \leq n\,\Delta t$, be the solution to

$$\begin{cases} \partial_t\, v + \partial_x\, f(v) = 0, \\ v(x, t = (n-1)\Delta t) = v(x,(n-1)\Delta t). \end{cases}$$

So $u(x,t)$ is the Riemann solution in the boxes $[(m-1)\Delta x,(m+1)\Delta x]\times[(n-1)\Delta t, n\,\Delta t]$.

Viewing the above construction, one may worry about several things:

One thing is that it is possible that this induction may fail at a stage N and the solution will defined only on $R^1\times(0,N\,\Delta t)$. That is, at stage N, $\left|u\left(a_{m-1}^{n-1}\right)-u\left(a_{m+1}^{n-1}\right)\right|$ may become so large that we cannot solve the Riemann problem uniquely. Even in each strip, is the solution well-defined?

The other one is that if the induction can be carried on to infinity, do we have the convergence of the sequence of approximate solutions? i.e. Can we have the stability of the scheme?

The third one is about the consistency of the scheme, i.e. if the approximate solutions converge, can the limit function be the weak entropy solution of the Cauchy problem?

Actually, Glimm solves these problems in his scheme:

a. In the space of functions of bounded total variation on R^1, the well-definedness and the stability are proved at the same time for suitably small initial data. So the BV norm estimate allows us to solve the Riemann problems step by step.

   However, no other satisfactory function space has been suggested until now to study weak solutions.

b. The stability estimate of Glimm gives strong compactness in $L^1_{loc}(R^1 \times R^1_+)$.

c. By the CFL condition, Glimm's approximate solution solves the equation exactly on each strip $R^1 \times ((n-1)\Delta t, n\,\Delta t)$. Thus for consistency, one has only to assess the error across $t = n\,\Delta t$. It is for this point that we require the randomness of mesh points.

(2) Glimm Functional and the stability of the scheme
Our first goal is to obtain the "BV" norm estimate on the approximate solutions.

For convenience of presentation, we need some terminologies.

a. "Diamond". For $m+n=$ odd (with $n>0$), the unique diamond centered at (x_m, t_n) is the region enclosed by the segments joining a^n_{m-1} to $a^{n\pm1}_m$ and $a^{n\pm1}_m$ to a^n_{m+1}. Here $x_m = m\,\Delta x$, $t_n = n\,\Delta t$. (See Figure 3.8)

   The advantage of using the notation of "diamond", is that the estimate on "Diamond" is easier to get. Then we can use it to approximate the "TV" estimate on the whole x-axis.

b. "Mesh curve", I-curve

- A mesh curve, I-curve, is an unbounded continuous, space like curve which consists of piecewise linear segments joining the mesh points a_m^n to a_{m+1}^{n+1} or a_{m+1}^{n-1} (but not both). (See Figure 3.9)

- For each $n \geq 0$, there is a unique I-curve, called J_n which connects all a_m^n to $a_{m\pm1}^{n+1}$ so that all the waves between t_n and t_{n+1} cross J_n.

  In particular, J_0 is the unique mesh curve which connects all the mesh points at $t = 0$. (See Figure 3.10)

- All the I-curves admit partial ordering: we say that J' precedes J, $J' < J$, if J lies toward later time.

  Two I-curves $J_- < J_+$, we say that J_+ is an "immediate successor" of J_- if the symmetric difference is a diamond. (See Figure 3.11)

c. Approaching waves on J

- Two elementary waves α_i, β_j across a mesh curve J (we denote this by $\alpha_i, \beta_j \epsilon J$), they are approaching if the waves on the left, say α_i, is the faster family compared with β_j, on the right, i.e. $i > j$; or if they are in the same family, then one of them has to be a shock. Denote the set of all pairs of approaching waves on J by $App(J)$ and set

$$N(J) = \sum_{App(J)} |\alpha_i| \cdot |\beta_j|.$$

  Then $N(J)$ takes into account of all the possible approaching waves in the future.

- Let Δ be a diamond, we say that two elementary waves α_i and β_j are approaching in Δ, if α_i, $\beta_j \in J_-$ but not on J_+, the immediate successor of J_-, and α_i, β_j are approaching on J_-. See Figure 3.12.

  Then we set
$$D(\Delta) = \sum_{App(\Delta)} |\alpha_i| \cdot |\beta_j|.$$

  $D(\Delta)$ will be used to measure the change of $N(J)$ from J_- to J_+.

d. Glimm's Functional

For a given I-curve J, we define a functional which is equivalent to the total variation of u across J as follows

$$L(J) = \sum_{\nu_j \,\epsilon\, J} |\nu_j|,$$

where the sum is taken over all the elementary waves across J.

In fact, this $L(J)$ might increase for later time. The increase is produced by wave interactions. However, if waves interact, they will not interact later. So the potential wave interaction functional $N(J)$ is decreasing.

Our aim is to choose a positive constant C large enough so that a new functional $G(J)$,

$$G(J) = L(J) + C\,N(J),$$

is decreasing.

Theorem 3.2 (Glimm) Assume that the Glimm scheme is defined up to mesh curve J_-. Then there exists a $\delta_0 > 0$, independent of J_- and Δt, such that as long as $L(J_-) \le \delta_0$, then

$$G(J_+) \le G(J_-),$$

where $J_+ > J_-$ is an immediate successor of J_-.

Proof Let Δ be the diamond between J_- and J_+. Let α and β be the left and right incoming waves to Δ. The ending waves leaving Δ is denoted by ε. Let

$$J_+ = J_0 \cup J'_+, \quad J_- = J_0 \cup J'_-.$$

We have

$$L(J_-) = L(J_0) + L(J'_-) = L(J_0) + \sum_{i=1}^n (|\alpha_i| + |\beta_i|)$$
$$L(J_+) = L(J_0) + L(J'_+) = L(J_0) + \sum_{i=1}^n |\varepsilon_i|$$

By the wave interaction estimates (Lemma 3.3), we have

$$\varepsilon_i = \alpha_i + \beta_i + D(\alpha, \beta)\,(1 + S(\alpha, \beta)),$$

where $D(\alpha, \beta) = \sum' |\alpha_j|\,|\beta_k|$ and the summation is over all approaching waves. $S(\alpha, \beta) = \max\{|\alpha_i|, |\beta_i|\}$.

Therefore, it follows that

$$\begin{aligned} L(J_+) - L(J_-) &= \sum_{i=1}^{n} (|\varepsilon_i| - (|\alpha_i| + |\beta_i|)) \\ &\leq \sum_{i=1}^{n} (|\alpha_i| + |\beta_i| - (|\alpha_i| + |\beta_i|) + D(\Delta)\,(1 + S(\alpha, \beta))) \\ &\leq D(\Delta)\,O(1). \end{aligned}$$

On the other hand, we have

$$N(J_+) = N(J_0) + N(J_0, J'_+),$$

where $N(J_0, J'_+)$ is the sum of the products of two approaching waves, one crossing J_0 and the other crossing J'_+. And

$$N(J_-) = N(J_0) + N(J'_-) + N(J_0, J'_-).$$

Note that

$$N(J_0, J'_+) = \sum{}' |\varepsilon_i|\,|\nu|,$$

where ν is any wave crossing J_0 such that ν, ε_i are approaching waves. Using Lemma 3.3 again, we claim that

$$\sum{}' |\varepsilon_i|\,|\nu| \leq \sum{}' (|\alpha_i| + |\beta_i|)\,|\nu| + O(1)\,D(\Delta)\,L(J_-). \qquad (3.29)$$

Actually, there is no problem for those terms that α_i, ν and β_i, ν are approaching waves. If ε_i and ν have the same index, and ν is a rarefaction wave, and if α_i (or β_i) is also a rarefaction wave, then it will not approach ν. However, in this case, we have $\varepsilon_i < 0$, $\alpha_i > 0$ (or $\beta_i > 0$). So from

$$\varepsilon_i = \alpha_i + \beta_i + O(1)\, D(\alpha, \beta)$$

it yields

$$\begin{array}{rcl} |\varepsilon_i| & < & |\beta_i + O(1)\, D(\alpha, \beta)| \\ (\text{or} \quad |\varepsilon_i| & < & |\alpha_i + O(1)\, D(\alpha, \beta)|\,) \end{array}$$

If α_i, β_i are both rarefaction wave, then $\alpha_i > 0, \beta_i > 0$, one has

$$|\varepsilon_i| < |O(1)\, D(\alpha, \beta)|.$$

Thus the claim (3.29) holds. So

$$\begin{aligned} N(J_0, J'_+) &\leq \sum^{1} (|\alpha_i| + |\beta_i|)\, |\nu| + O(1)\, D(\Delta)\, L(J_-) \\ &\leq N(J_0, J'_-) + O(1)\, D(\Delta)\, L(J_-). \end{aligned}$$

$$\begin{aligned} N(J_+) - N(J_-) &\leq -N(J'_-) + O(1)\, D(\Delta)\, L(J_-) \\ &= -D(\Delta) + O(1)\, D(\Delta)\, L(J_-). \end{aligned}$$

By definition,

$$G(J_-) = L(J_-) + C\,N(J_-),$$
$$G(J_+) = L(J_+) + C\,N(J_+),$$

therefore, one has

$$\begin{aligned} G(J_+) - G(J_-) &= L(J_+) - L(J_-) + C(N(J_+) - N(J_-)) \\ &\le O(1)\,D(\Delta) - C\,D(\Delta) + C\,O(1)\,D(\Delta)\,L(J_-) \\ &= C\,D(\Delta)\left[-1 + \frac{O(1)}{C} + O(1)\,L(J_-)\right]. \end{aligned}$$

Choose δ_0, C such that $O(1)\,\delta_0 \le \frac{1}{2}$, $\frac{O(1)}{C} \le \frac{1}{4}$, one has

$$G(J_+) - G(J_-) \le 0.$$

Theorem 3.3 There exists a positive constant $\delta_1 > 0$ such that if $L(J_0) \le \delta_1$. Then the Glimm scheme can be defined for all time and for any I-curve J. Furthermore, we have

$$L(J) \le 2\delta_1.$$

Proof From Theorem 3.2, we know that if J_0^+ is an immediate successor of J_0, then there exists a $C > 0$ such that

$$L(J_0^+) + C\,N(J_0^+) \le L(J_0) + C\,N(J_0) \le L(J_0) + C\,L^2\,(J_0).$$

So if $L(J_0) < \min\left\{1, \frac{1}{C}\right\}$, then

$$L(J_0^+) + C\,N(J_0^+) \le 2L(J_0)$$

Thus if $L(J_0)$ is small, the Glimm scheme can be defined on J_0^+.

Now, by induction, for any I-curve $J > J_0$, we can start from J_0 to J by immediate successors and we have

$$L(J) + C\,N(J) \leq L(J_0) + C\,N(J_0) \leq 2L(J_0).$$

Hence, there exists a small positive constant $\delta_1 > 0$ such that if $L(J_0) \leq \delta_1$, which is equivalent to the fact that the total variation of u_0 is small, then

$$L(J) \leq 2\delta_1, \quad \forall\, J > J_0. \tag{3.30}$$

At the same time, the inequality (3.30) guarantees the Glimm scheme can be defined for all time and for any I-curve J.

The proof of the theorem is finished.

We denote the approximate solutions constructed through Glimm Scheme by $u_\theta^{\Delta t}$ or $u_\theta^{\Delta x}$. Then as a consequence of Theorem 3.2 and Theorem 3.3, we have shown that

Corollary 3.1 There exists a $\delta > 0$ such that if $TV\ u_0 \leq \delta$, then
(1) $OSC\ u_\theta^{\Delta t} \leq TV\ u_\theta^{\Delta t} \leq C_1\ TV\ u_0$;
(2) $\sup u_\theta^{\Delta t} \leq C_2$,
where C_1 and C_2 are some constants.

Corollary 3.2 (Temperal estimates) Under the same assumption in Theorem 3.3, one has that for any $t, t' > 0$,

$$\int_{-\infty}^{+\infty} |u_\theta^{\Delta t}(x,t) - u_\theta^{\Delta x}(x,t')|dx \leq C_3 |t - t'|,$$

where C_3 is independent of t and t'.

Proof For any fixed t, t', we assume $t' > t$ without loss of generality. Let

$$D(x,t') = \left\{(y,t)|\, |y - x| \leq \frac{\Delta x}{\Delta t}(t' - t)\right\}.$$

(See Figure 3.14)

Due to CFL conditions (3.28), it concludes that $D(x,t)$ contain the domain of dependence of (x,t'). Now define

$$V(y,t) = \begin{cases} u_\theta^{\Delta x}(y,t), & (y,t) \epsilon D(x,t'), \\ \bar{u}_+ = \lim_{y \to (x + \frac{\Delta x}{\Delta t}(t'-t))-} u_\theta^{\Delta x}(y,t), & y \geq x + \frac{\Delta x}{\Delta t}(t'-t), \\ \bar{u}_- = \lim_{y \to (x - \frac{\Delta x}{\Delta t}(t'-t))+} u_\theta^{\Delta x}(y,t), & y \leq x - \frac{\Delta x}{\Delta t}(t'-t). \end{cases}$$

Denote by $V_\theta^{\Delta x}(y,t)$ the Glimm approximate solution with Cauchy data $V(y,t)$. Then, it is clear that

$$u_\theta^{\Delta x}(x,t) = V_\theta^{\Delta x}(x,t)$$

and since $D(x,t')$ contain the domain of depence of (x,t'), one has

$$u_\theta^{\Delta x}(x,t') = V_\theta^{\Delta x}(x,t')$$

Furthermore, one has

$$\lim_{y\to+\infty} V_\theta^{\Delta x}(y,t) = \lim_{y\to+\infty} V_\theta^{\Delta x}(y,t') = \bar{u}_+,$$

$$\lim_{y\to-\infty} V_\theta^{\Delta x}(y,t) = \lim_{y\to-\infty} V_\theta^{\Delta x}(y,t') = \bar{u}_-.$$

It follows that

$$\begin{aligned}
& \left|u_\theta^{\Delta x}(x,t') - u_\theta^{\Delta x}(x,t)\right| = \left|V_\theta^{\Delta x}(x,t') - V_\theta^{\Delta x}(x,t)\right| \\
\le\ & \left|V_\theta^{\Delta x}(x,t') - \bar{u}_+\right| + \left|V_\theta^{\Delta x}(x,t) - \bar{u}_+\right| \\
\le\ & TV\ V_\theta^{\Delta x}(\cdot,t') + TV\ V_\theta^{\Delta x}(\cdot,t) \\
\le\ & O(1)\ TV\ V_\theta^{\Delta x}(\cdot,t) \qquad \text{(by Theorem 3.3)} \\
=\ & O(1)\ TV\ u_\theta^{\Delta x}(\cdot,t)|_{D(x,t')}
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \left|u_\theta^{\Delta x}(x,t') - u_\theta^{\Delta x}(x,t)\right| \, dx \le C_4 \int_{-\infty}^{+\infty} \; T.V.\, u_\theta^{\Delta x}(\cdot,t)\Big|_{D(x,t')} \, dx \\
= \; & C_4 \int_{-\infty}^{+\infty} \left(\int_{x-O(1)\,(t'-t)}^{x+O(1)\,(t'-t)} |\, d\, u_\theta^{\Delta x}(\cdot,t)| \right) dx \\
= \; & C_4 \int_{-\infty}^{+\infty} \left(\int_{-O(1)\,(t'-t)}^{O(1)\,(t'-t)} |\, d\, u_\theta^{\Delta x}(x+\cdot,t)| \right) dx \\
= \; & C_4 \int_{-O(1)\,(t'-t)}^{O(1)\,(t'-t)} \int_{-\infty}^{+\infty} |d\, u_\theta^{\Delta x}(x+\cdot,t)| \, dx \\
= \; & O(1)|t'-t| \; TV \; u_\theta^{\Delta x}(\cdot,t) \\
\le \; & C|t'-t| \; TV \; u_0 \qquad \text{(by Corollary 3.1)}
\end{aligned}$$

Theorem 3.4 (Compactness of Glimm Solution)
There exists a subsequence of $\{u_\theta^{\Delta x} : \theta \epsilon \Phi, \Delta t > 0\}$, which converges in L^1_{loc} to a function $u(x,t)$. Furthermore, $u(x,t)$ satisfies

(i) $||u(\cdot,t)||_{L^\infty} \leq C_1$;

(ii) $T.V.\,u(\cdot,t) \leq C_2$;

(iii) $||u(\cdot,t_1) - u(\cdot,t_2)||_{L^1_{loc}} \leq C_3\,|t_2 - t_1|$,

where $C_i\,(i = 1,2,3)$ are constants.

Proof By our previous estimates, we have

(H_1) $||u_\theta^{\Delta x}(\cdot,t)||_{L^\infty} \leq C_1,$

(H_2) $T.V.\, u_\theta^{\Delta x}(\cdot,t) \leq C_2,$

(H_3) $||u_\theta^{\Delta x}(\cdot,t_2) - u_\theta^{\Delta x}(\cdot,t_1)||_{L^1_{loc}} \leq C_3\,|t_2 - t_1|$

By (H_1) and (H_2), use Helley principle to get a countable set $\{t_m\} \subset [0,T]$, where $\{t_m\}$ is dense in $[0,T]$, such that $\{u_\theta^{\Delta x_i}(x,t)\}$ converges at any point on each line $t = t_m$ $(m = 1, 2, \cdots)$ as $\Delta x_i \to 0^+$ ($\Delta t_i \to 0^+$, by CFL). We still denote $u_\theta^{\Delta x_i}$ by u_i. It is noted that $\Delta x_i \to 0^+$ as $i \to +\infty$. We will show that u_i converges in $L^1_{loc}(R^1 \times R^1_+)$ or $L^1_{loc}(R^1 \times (0,T))$ for all $T > 0$. For this purpose, we will show that for any $X > 0$,

$$I_{ij}\,(t) = \int_{-X}^{X} |u_j(x,t) - u_i(x,t)|dx \to 0 \text{ as } i,j \to +\infty \text{ for a.e. } t\,\epsilon\,[0,\,T],$$

i.e. $u_i(x,t)$ forms a Cauchy sequence in $L^1(|x| \leq X)$.

For any given $t\,\epsilon\,[0,\,T]$, there exists a $\{t_{m'}\} \subset \{t_m\}$ such that $t_{m'} \to t$ as $m' \to +\infty$. Then

$$\begin{aligned} I_{ij}\,(t) \quad &\leq \quad \int_{-X}^{X} |u_j(x,t) - u_j(x,t_{m'})|dx + \int_{-X}^{X} |u_j(x,t_{m'}) - u_i(x,t_{m'})|dx \\ &\quad + \int_{-X}^{X} |u_i(x,t_{m'}) - u_i(x,t)|dx \\ &\leq \quad \int_{-X}^{X} |u_j(x,t_{m'}) - u_i(x,t_{m'})|dx + 2C_3|t_{m'} - t| \qquad (\text{ by } (H_3)) \end{aligned}$$

Note that $\{u_i(x, t_{m'})\}$ is a Cauchy sequence in $L^1(|x| \le X)$, we obtain that for any $\varepsilon > 0$, we first choose m' large enough such that $2C_3|t_{m'} - t| < \frac{\varepsilon}{2}$, then choose i, j large enough such that

$$\int_{-X}^{X} |u_j(x, t_{m'}) - u_i(x, t_{m'})| dx < \frac{\varepsilon}{2}.$$

This proves that

$$I_{ij}(t) \to 0 \text{ as } i, j \to +\infty.$$

We have that $\{u_i(x,t)\}$ is a Cauchy sequence in $L^1_{loc}(R^1 \times R^1_+)$. We denote the limit by $u(x,t)$. Then there exists a subsequence of $\{u_i(x,t)\}$ still denoted by itself such that

$$u_i(x,t) \to u(x,t) \text{ a.e. } (x,t) \in R^1 \times R^1_+.$$

And (i), (ii), (iii) of the theorem can be obtained from (H_1), (H_2) and (H_3). The proof of the theorem is finished.

§3.4 Consistency of Glimm scheme

Up to now, we have proved all the things except that $u(x,t)$ is a weak solution. To show that $u(x,t)$ gives a weak solution, we have to assess the error due to $u_i = u_\theta^{\Delta x_i}(x,t)$. Recall that the approximate sequence $\{u_\theta^{\Delta x}(x,t)\}$ has the following properties:

(i) $|u_\theta^{\Delta x}(\cdot,t)|_{L^\infty} \le M_1$

(ii) $TV\, u_\theta^{\Delta x}(\cdot,t) \le M_2 = C_1 \cdot TV\, u_0$

(iii) $\int_{|x|\le R} |u_\theta^{\Delta x}(\cdot,t_1) - u_\theta^{\Delta x}(\cdot,t_2)|dx \le C_R \cdot |t_2 - t_1| \qquad \forall\, R > 0$

Then $u_\theta^{\Delta x}(x,t) \to u(x,t)$ a.e. as $\Delta x \to 0$ for any $\theta \,\epsilon\, \Theta = \prod[-1,1]$, $t > 0$. Let

$$\mathcal{E}_{\varphi}(u, f(u)) = \iint_{R^1 \times R^1_+} \partial_t \varphi \cdot u + \partial_x \varphi \cdot f(u)\, dx\, dt + \int_{R^1} \varphi(x,0)\, u(x,0)\, dx$$

The ideal situation in the proof is that for any $\varphi \,\epsilon\, C_c^1\,(R^1 \times R^1_+)$, $\theta \,\epsilon\, \Theta$, we want to get $\mathcal{E}_{\varphi}(u_i, f(u_i)) = \mathcal{E}_{\varphi}(u_\theta^{\Delta x_i}, f(u_\theta^{\Delta x_i})) \to 0$ as $\Delta x_i \to 0^+$. Unfortunately, this ideal situation is false for some several $\theta \,\epsilon\, \Theta$. Readers can see the example in the book of Smoller. To conquer this, we may take over all θ to be random in Θ.

We compute $\mathcal{E}_{\varphi}(u_\theta^{\Delta x}, f(u_\theta^{\Delta x}))$ directly. From the construction by Glimm scheme, on each time interval $((n-1)\,\Delta t, n\,\Delta t)$, $u_\theta^{\Delta x}$ solves the Riemann problem. Hence

$$\begin{aligned}
& \mathcal{E}_{\varphi}(u_{\theta}^{\Delta x}, f(u_{\theta}^{\Delta x})) \\
= \; & \mathcal{E}(u_{\theta}^{\Delta x}, f(u_{\theta}^{\Delta x}), \varphi) \\
= \; & \sum_{n=1}^{\infty} \iint_{R^1 \times ((n-1)\Delta t, n\Delta t)} (\partial_t \varphi \cdot u_{\theta}^{\Delta x} + \partial_x \varphi \cdot f(u_{\theta}^{\Delta x})) dx\, dt \\
& + \int_{R^1} \varphi(x, t=0)\, u_{\theta}^{\Delta x}(x, t=0)\, dx \\
= \; & \sum_{n=1}^{\infty} \int_{R^1} \varphi(x,t)\, u_{\theta}^{\Delta x}(x,t) \Big|_{t=(n-1)\,\Delta t+}^{t=n\,\Delta t-} dx \\
& + \int_{R} \varphi(x, t=0)\, u_{\theta}^{\Delta x}(x, t=0)\, dx \\
= \; & -\sum_{l=1}^{\infty} J_l(\theta, x, \varphi)
\end{aligned}$$

where

$$
\begin{aligned}
J_l = J_l(\theta, \Delta x, \varphi) &= \int_{R^1} \left(u_\theta^{\Delta x}(x, l\,\Delta t+) - u_\theta^{\Delta x}(x, l\,\Delta t-) \right) \cdot \varphi(x, l\,\Delta t) dx \\
&= \int_{R^1} \left[u_\theta^{\Delta x}(x, l\,\Delta t) \right] \varphi(x, l\,\Delta t) dx
\end{aligned}
$$

$$
\left[u_\theta^{\Delta x}(x, l\,\Delta t) \right] = u_\theta^{\Delta x}(x, l\,\Delta t+) - u_\theta^{\Delta x}(x, l\,\Delta t-)
$$

We denote
$J(\theta, \Delta x, \varphi) = -\sum_{l=1}^{\infty} J_l(\theta, \Delta x, \varphi) = \mathcal{E}\left(u_\theta^{\Delta x},\, f(u_\theta^{\Delta x}),\, \varphi\right)$. First, we start with a rough estimate on $J(\theta, \Delta x, \varphi)$.

Lemma 3.4 There exist $M,\ M_1 > 0$ independent of $\varphi, \Delta x, \theta$ such that

(a) $|J_l(\theta, \Delta x, \varphi)| \leq M\,\Delta x \cdot ||\varphi||_{L^\infty} \quad \forall \quad l = 1, 2, \cdots$

(b) $|J(\theta, \Delta x, \varphi)| \leq M_1\, diam(supp \varphi)\, ||\varphi||_{L^\infty}$

here $diam\,(supp\,\varphi) = \sup\{|x - y| + |t - \tau|\ : (x, t), (y, \tau)\,\epsilon\, supp\,\varphi\}$

Proof: (b) is a consequence of (a). Let $D > 0$ be such that $\varphi(x, t) = 0\ \forall\, x\,\epsilon\, R,\ t > D$, and $\Lambda = \frac{\Delta t}{\Delta x} \leq C$ by CFL condition. Then

$$
\begin{aligned}
|J(\theta, \Delta x, \varphi)| &\leq \sum_{l=1}^{\infty} |J_l(\theta, \Delta x, \varphi)| \\
&= \sum_{l=1}^{D/\Delta t} |J_l(\theta, \Delta x, \varphi)| \\
&\leq M \, \Delta x \, ||\varphi||_{L^\infty} \cdot \frac{D}{\Delta t} \\
&= \frac{M}{\Lambda} \cdot D \, ||\varphi||_{L^\infty}
\end{aligned}
$$

So it suffices to prove (a). To do this, since $u_\theta^{\Delta x}(x, t)$ solve the Riemann problem in the region $((m-1)\,\Delta x, (m+1)\,\Delta x) \times ((l-1)\,\Delta t, l\,\Delta t)$ with $m + l =$ even, we have

$$
\begin{aligned}
& |J_l(\theta, \Delta x, \varphi)| \\
\leq \quad & \int_R |[u_\theta^{\Delta x}(x, l\,\Delta t)]| \cdot |\varphi(x, l\,\Delta t)| dx \\
= \quad & \sum_{m+l=even} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |[u_\theta^{\Delta x}(x, l\,\Delta t)]| \cdot |\varphi(x, l\,\Delta t)| dx \\
= \quad & \sum_{m+l=even} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |\varphi(x, l\,\Delta t)| \cdot |u_\theta^{\Delta x}(x, l\,\Delta t+) - u_\theta^{\Delta x}(x, l\,\Delta t-)| dx \\
= \quad & \sum_{m+l=even} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |\varphi(x, l\,\Delta t)| \cdot |u_\theta^{\Delta x}((m+\theta_l)\Delta x, l\,\Delta t-) - u_\theta^{\Delta x}(x, l\,\Delta t-)| dx \\
\leq \quad & ||\varphi||_{L^\infty} \sum_{m+l=even} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |u_\theta^{\Delta x}((m+\theta_l)\Delta x, l\,\Delta t-) - u_\theta^{\Delta x}(x, l\,\Delta t-)| dx \\
\leq \quad & ||\varphi||_{L^\infty} \sum_{m+l=even} TV_{[(m-1)\Delta x, (m+1)\Delta x]}\, u_\theta^{\Delta x}(\cdot, l\,\Delta t-) \cdot 2\,\Delta x \\
= \quad & 2\,\Delta x \cdot ||\varphi||_{L^\infty}\ TV\ u_\theta^{\Delta x}(\cdot, l\,\Delta t-) \\
\leq \quad & 2\,M_2 \cdot \Delta x\, ||\varphi||_{L^\infty}
\end{aligned}
$$

where M_2 is stated in (ii).

The estimate is too rough to show $J(\theta, \Delta x, \varphi) \to 0$ as $\Delta x \to 0$. Now we regard $\theta \,\epsilon\, \Theta$ as a random variable. To describe this precisely, we set $\Theta = \Pi\,[-1, 1] \approx \Pi\,[0, 1]$ so that Θ becomes a probability space. Our goal is to show that there is a null set $N \subset \Theta$ $(meas(N) = 0)$ such that for any $\theta \,\epsilon\, \Theta \backslash N$, and $\varphi \,\epsilon\, C_c^1$, $J(\theta, \Delta x, \varphi) \to 0$ as $\Delta x \to 0^+$. To this end, we need one more lemma.

Lemma 3.5 Suppose φ is piecewise constant on each segment $[(m-1)\Delta x, (m+1)\Delta x] \times \{l\,\Delta t\}, m+l = even$. Then

$$J_{l_1}(\cdot, \Delta x, \varphi) \perp J_{l_2}(\cdot, \Delta x, \varphi) \text{ on } L^2(\Theta) \text{ if } l_1 \neq l_2$$

Proof: The main idea is that independent random variable with zero mean are orthogonal, that is, we go to prove that

(1) If $l_1 < l_2$, then J_{l_1} is independent of θ_{l_2}.

(2) $\int_\Theta J_l \, d\theta = 0$.

Indeed, (1) follows by definition of the Glimm scheme. For I_1, J_{I_1} depends only on the construction before time, and does not depend on the random variable θ_{I_2} after time. To show (2), from

$$\int_{\Theta} J_I\,(\theta, \Delta x, \varphi)\,d\theta = \int \left(\int J_I(\theta, \Delta x, \varphi)\,d\theta_I \right)\,d\tilde{\theta}$$

here $d\tilde{\theta} = \prod_{j \neq I} d\theta_j$. It suffices to compute $\int J_I(\theta, \Delta x, \varphi)\,d\theta_I$. From similar computation as before,

$$
\begin{aligned}
&\int_{-1}^{1} J_l(\theta, \Delta x, \varphi)\, d\theta_l \\
= &\int_{-1}^{1} \sum_{m+l=even} \int_{(m-1)\Delta x}^{(m+1)\Delta x} \varphi(x, l\,\Delta t)\,(u_\theta^{\Delta x}\,((m+\theta_l)\Delta x, l\,\Delta t-) \\
&\qquad\qquad -u_\theta^{\Delta x}\,(x, l\,\Delta t-))dx\, d\theta_l \\
= &\sum_{m+l=even} C_{\varphi,m,l} \int_{-1}^{1}\int_{(m-1)\Delta x}^{(m+1)\Delta x} u_\theta^{\Delta x}\,((m+\theta_l)\Delta x, l\,\Delta t-) \\
&\qquad\qquad -u_\theta^{\Delta x}\,(x, l\,\Delta t-) dx\, d\theta_l \qquad (3.31)
\end{aligned}
$$

Now we claim that the right hand side of (3.31) is zero. To do this, since $u_\theta^{\Delta x}(x, l\,\Delta t-)$ depends only on θ_i, $i = 0, 1, \cdots, l-1$, and does not depend on θ_l, we have

$$\int_{-1}^{1}\int_{(m-1)\Delta x}^{(m+1)\Delta x} u_\theta^{\Delta x}(x, l\,\Delta t-)dx\,d\theta_l = 2\int_{(m-1)\Delta x}^{(m+1)\Delta x} u_\theta^{\Delta x}(x, l\,\Delta t-)dx$$

also,

$$\begin{aligned} & \int_{-1}^{1}\int_{(m-1)\Delta x}^{(m+1)\Delta x} u_\theta^{\Delta x}((m+\theta_l)\Delta x, l\,\Delta t-)dx\,d\theta_l \\ = & \int_{-1}^{1} u_\theta^{\Delta x}((m+\theta_l)\Delta x, l\,\Delta t-)\,d\theta_l \cdot 2\Delta x \\ = & 2\int_{(m-1)\Delta x}^{(m+1)\Delta x} u_\theta^{\Delta x}(y,\, l\,\Delta t-)dy \end{aligned}$$

Hence the claim holds and $\int J_l(\theta, \Delta x, \varphi)\, d\theta = 0$. Now for $l_1 \neq l_2$, say $l_1 < l_2$, by (1) and (2), we deduce that

$$\begin{aligned}
< J_{l_1}, J_{l_2} > &= \int_\Theta J_{l_1}(\theta, \Delta x, \varphi) \cdot J_{l_2}(\theta, \Delta x, \varphi) d\theta \\
&= \int \left(\int J_{l_1}(\theta, \Delta x, \varphi) \cdot J_{l_2}(\theta, \Delta x, \varphi) d\theta_{l_2} \right) \Pi_{l \neq l_2}\, d\, \theta_l \\
&= \int J_{l_1} \cdot \left(\int J_{l_2}\, d\theta_{l_2} \right) \Pi_{l \neq l_2}\, d\, \theta_l \\
&= 0
\end{aligned}$$

This proves Lemma 3.5.

Lemma 3.5 means merely that we can ignore all the intersection terms $J_{l_1} \cdot J_{l_2}$ for $l_1 \neq l_2$. We can ready to state the main consistency theorem. This theorem completes the theory of Glimm scheme.

Theorem 3.5 There exists a null set $N \subset \Theta$ and a sequence $\Delta x_i \to 0$ such that for any $\theta \,\epsilon\, \Theta \setminus N$ and any $\varphi \,\epsilon\, C_c^1\,(t > 0)$,

$$J(\theta, \Delta x_i, \varphi) \to 0 \text{ as } \Delta x_i \to 0$$

Proof:

Step 1: Let φ satisfies the condition in Lemma 3.5. Then

$$\begin{aligned} ||J(\cdot,\Delta x,\varphi)||^2_{L^2(\Theta)} &= \sum_{l=1}^{\infty} ||J_l(\cdot,\Delta x,\varphi)||^2_{L^2(\Theta)} \\ &\leq \sum_{l=1}^{\infty} ||J_l(\cdot,\Delta x,\varphi)||^2_{L^\infty(\Theta)} \\ &\leq M^2 \sum_{l\in\Lambda} (\Delta x_i)^2 \, ||\varphi||^2_{L^\infty} \\ &\leq \bar{M}\, \Delta x_i \, diam\,(supp\,\varphi)\, ||\varphi||^2_{L^\infty} \end{aligned}$$

where $\Lambda = \{l : R^1 \times \{l\,\Delta t\} \cap supp\,\varphi \neq \phi\}$. The first equality is due to Lemma 3.5, the second line is due to the probability measure on Θ, the third line comes by Lemma 3.4. Thus $J(\cdot, \Delta x_i, \varphi) \to 0$ as $\Delta x_i \to 0^+$ in $L^2(\Theta)$. Therefore, there is a null set N_φ depending on φ with $meas(N_\varphi) = 0$ such that $J(\cdot, \Delta x_i, \varphi) \to 0$ as $\Delta x_i \to 0$ for all $\theta \,\epsilon\, \Theta \setminus N_\varphi$.

Step 2: For any $\varphi \,\epsilon\, L_c^\infty$, by Lemma 3.4 (b), we have

$$\begin{aligned} ||J(\cdot, \Delta x, \varphi)||_{L^2(\Theta)} &\leq ||J(\cdot, \Delta x, \varphi)||_{L^\infty(\Theta)} \\ &\leq C\,||\varphi||_{L^\infty} \end{aligned}$$

Step 3: Let φ_ν be a sequence of piecewise constant function with compact support which is L^∞ and dense in C_c^1. For each φ_ν, by step 1, there is a null set $N_\nu \subset \Theta$ and a subsequence $\Delta x_{i_k} \to 0$ such that $J(\theta, \Delta x_{i_k}, \varphi_\nu) \to 0$ as $\Delta x_{i_k} \to 0 \quad \forall\, \theta \,\epsilon\, \Theta \setminus N_\nu$. Set $N = \cup_{\nu=1}^{\infty} N_\nu$ and choose a subsequence Δx_i (by diagonal process) such that for any ν, $J(\theta, \Delta x_i, \varphi_\nu) \to 0$ as $\Delta x_i \to 0 \quad \forall\, \theta \,\epsilon\, \Theta \setminus N$.

For any $\varphi \epsilon C_c^1$, choose a sequence of piecewise constant function $\varphi_{\nu_k} \epsilon L_c^\infty$ as above such that $||\varphi_{\nu_k} - \varphi||_{L^\infty} \to 0$ as $\nu_k \to +\infty$. Hence

$$\begin{aligned} |J(\theta, \Delta x_i, \varphi)| &\leq |J(\theta, \Delta x_i, \varphi - \varphi_{\nu_k})| + |J(\theta, \Delta x_i, \varphi_{\nu_k})| \\ &\leq C\,||\varphi - \varphi_{\nu_k}||_{L^\infty} + |J(\theta, \Delta x_i, \varphi_{\nu_k})| \end{aligned}$$

and tends to zero by first choosing φ_{ν_k} so that $C\,||\varphi - \varphi_{\nu_k}||_{L^\infty} < \frac{\varepsilon}{2}$, then choosing Δx_i small such that $|J(\theta, \Delta x_i, \varphi_{\nu_k})| < \frac{\varepsilon}{2}$. This proves the theorem.

§3.5 Front Tracking Method

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \quad u \epsilon \mathbb{R}^n, \quad x \epsilon \mathbb{R}, \quad t > 0 \\ u(x, t = 0) = u_0(x) \end{cases} \tag{3.32}$$

Assumption:

(i) f is smooth in Ω.

(ii) each characteristic family is either genuinely nonlinear or linearly degenerate.

Approximate solution by front tracking:

Step 1: Construct u_0^δ such that

1. u_0^δ is piecewise constant with finite many jumps.
2. $T.V.u_0^\delta \leq T.V.u_0$
3. $\int |u_0^\delta - u_0| dx \to 0 \qquad \text{as} \qquad \delta \to 0^+$

Step 2: Resolving the initial jump by solving Riemann problems

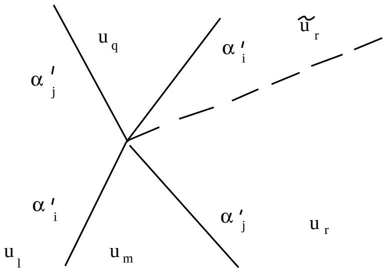
Caution If one uses this Riemann solver, one might find the number of interactions could go to infinity at finite time, so that one cannot extend the solution globally (due to the complexity of the wave interaction in system).

Idea If the scheme is stable in BV, the most of the new waves are extremely small, thus, can be ignored.

Simplified Riemann Solver

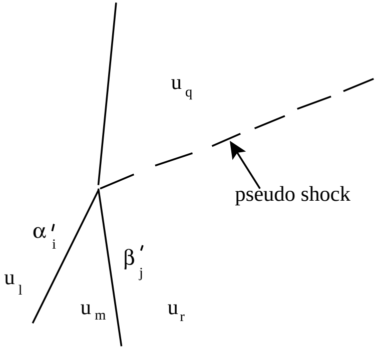
Case 1 $i > j$

$$\begin{aligned} u_m &= T_i(\alpha_i', u_l) \\ u_r &= T_j(\alpha_j', u_m) \\ u_q &= T_j(\alpha_j', u_l) \\ \tilde{u}_r &= T_i(\alpha_i', u_q) \end{aligned}$$



Case 2 $i = j$

$$\begin{aligned} u_m &= T_i(\alpha_i', u_l) \\ u_r &= T_i(\beta_i', u_m) \\ u_q &= T_i(\alpha_i' + \beta_i', u_l) \end{aligned}$$



Case 3 In case of a front with a pseudo shock

$$u_q = T_i(\alpha_i', u_l)$$
(u_q, u_r) forms a pseudo shock with speed $\hat{\lambda}$.

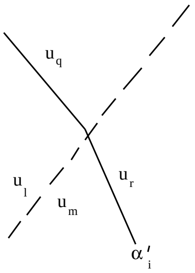
(1.) $t = 0$. Accurate Riemann Solver (ARS).

(2.) at an interaction $\tilde{t} > 0$, the two incoming fronts, α, β

if $|\alpha|\,|\beta| > \sigma$, use ARS;

otherwise $|\alpha|\,|\beta| < \sigma$, use SRS.

(3.) at the interaction time $\tilde{t}$ which involves pseudo shocks, use SRS.



§3.6 A Front Tracking Algorithm

1. Accurate Riemann Solver

Let α interact with β to produce a solution $\xi = (\varepsilon_1, \cdots, \varepsilon_n)$. If all ε_i is either shock or contact discontinuity, leave it alone. Otherwise $\varepsilon_i > 0$, as the i-th wave is center rarefaction wave. Then we divide this rarefaction wave into small fan of discontinuity in the following way:

$$\text{For given} \quad \delta > 0, \quad \text{let} \quad \nu = \left[\frac{\varepsilon_i}{\delta}\right]$$

Assume the i-wave is

$$u_T = T_i(\varepsilon_\nu, u_-) = R_i(\varepsilon_i, u_-), \quad \varepsilon_i > 0$$

$$\begin{aligned} \text{Let} \quad & u^0 = u_-, & u^j = R_i(j\delta, u_-), & \qquad j = 1, \cdots, \nu \\ & u^{\nu+1} = u_+ & & \\ & u^\varepsilon(x,t) = u^j & \text{when} \qquad & \lambda_i(u^j) < \tfrac{x}{t} < \lambda_i(u^{j+1}) \end{aligned}$$

2. Simplified Riemann Solver (SRS)

3. Implement

Step 1: Let u_0^δ be a piecewise constant approximation of $u_0(x)$ with N-jumps ($N < \infty$) such that

$$\begin{cases} T.V.\ u_0^\delta & \leq & T.V.\ u_0 \\ \int |u_0 - u_0^\delta| dx & \leq & \delta \end{cases}$$

Then apply ARS to u_0^δ.

Step 2: When they interact, we first specify a constant $\sigma > 0$. Let the interacting fronts be α and β. Then we will use

$$\begin{aligned} \text{ARS} \quad \text{if} \quad |\alpha| \cdot |\beta| \geq \sigma \\ \text{SRS} \quad \text{if} \quad |\alpha| \cdot |\beta| < \sigma \end{aligned}$$

(Here and from now on, front mean either shocks or rarefaction front or contact discontinuity.)

If one of the incoming waves is a pseudo shock, then we will use SRS always. Since we will show that total amount of pseudo shock is small.

Order of waves:

Definition 3.8 The generation order of a wave is the maximum number of collisions predating its birth.

Remark: All the waves presenting at $t = 0$ has order $\mu = 0$. All the new waves produced by wave interactions, say ε is a new wave which is produced by interaction of α and β with order μ_1 and μ_2, $O(\alpha) = \mu_1$, $O(\beta) = \mu_2$.

Case 1: α and β are in different family, α is i-family, β is in the j-family, ε is in the k-family.

$$\begin{array}{lll} \text{if} & k = i, & O(\varepsilon) = \mu_1, \\ \text{if} & k = j, & O(\varepsilon) = \mu_2, \\ \text{if} & k \neq i, j, & O(\varepsilon) = \max\{\mu_1, \mu_2\} + 1 \end{array}$$

Case 2: α and β are in the same family, i-family.

$$\begin{array}{lll} \text{if} & k = i, & O(\varepsilon) = \min\{\mu_1, \mu_2\}, \\ \text{if} & k \neq i, & O(\varepsilon) = \max\{\mu_1, \mu_2\} + 1 \end{array}$$

Approximate Characteristics: $X_i(t)$ is said to be an i-characteristic if $X_i(t)$ is a piecewise line segment with constant slope $\lambda_i(u^\delta)$ if u^δ is constant and becomes an i-front when it hits an i-front.

§3.7 Approximate Solution

Goal: Eventually, we need to show the previously constructed scheme produces a "good" approximate solution.

Definition 3.9 (Approximate solution) For any $\varepsilon > 0$. An ε-approximate solution to the Cauchy problem (1) is a vector-valued piecewise constant function separated by finitely many line segments with the following properties:

1. Each wave may originate from either $t = 0$ or at the collision points of two other waves and the wave in general will stay forever unless it collides with other waves.
2. There are finitely many collision points.

3. All the waves are classified into three classes
   (1.) shock wave or contact discontinuity: i-shock or i-contact discontinuity is a triple $(u_l,\ u_r,\ x(t))$ such that $u_r = s_i(\varepsilon_i, u_l)$ and $|\dot{x}_i - s_i| \leq \delta$ (where s_i is the speed of original shock or contact discontinuity).
   (2.) Rarefaction front: an i-rarefaction front is a triple $(u_l,\ u_r,\ \dot{x}_i)$ such that $u_r = R_i(\tau, u_l)$, $0 < \tau < \delta$, and

$$|\dot{x}_i - \lambda_i(u_r)| \leq \delta$$

   (3.) Pseudo-shock: a pseudo shock is a triple $(u_l,\ u_r,\ \lambda_{n+1}t)$ is a discontinuity travelling with speed λ_{n+1} .

$$\sum_{y \epsilon ps} |u(y(t)+, t) - u(y(t)-, t)| \leq \delta$$

4. $\displaystyle\int_{-\infty}^{\infty} |u^{\delta}(x, 0) - u_0(x)| dx \leq \delta$

Theorem 3.6 The front tracking algorithm discussed before indeed produce a δ-approximation solution if one chooses δ and σ appropriately and $T.V.u_0$ is small.

Sketch of idea of the proof

1. Estimate of $u^\delta(x,t)$:
   - scheme has to be stable,
   - to avoid produce too many fronts.

   Glimm's idea is crucial.
2. Total amount of pseudo shocks $\leq \delta$:
   - tracking the order of waves.

(1.) interaction estimate

(2.) Glimm functional

Proof of Theorem 3.6

Step 1 Wave interaction estimates.

Lemma 3.6 (ARS) i-wave α and j-wave β interact and then produce waves $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$.

Case 1 $i > j$. $$|\varepsilon_i - \alpha| + |\varepsilon_j - \beta| + \sum_{k \neq i,j} |\varepsilon_k| = O(1)|\alpha| \cdot |\beta|$$

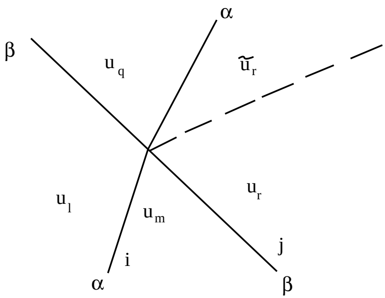
Case 2 $i = j$. $$|\varepsilon_i - (\alpha + \beta)| + \sum_{k \neq i} |\varepsilon_k| = O(1)|\alpha| \cdot |\beta|$$

Lemma 3.7 (SRS)

<u>Case 1</u> $|\alpha| \cdot |\beta| < \sigma$.

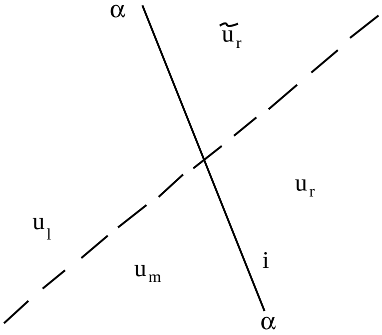
$$|\tilde{u}_r - u_r| = O(1)|\alpha| \cdot |\beta|$$

(whether $i = j$ or not, we always have the above estimate)



Case 2 One pseudo shock interacts with one front, then

$$|\tilde{u}_r - u_r| - |u_m - u_l| = O(1)|\alpha| \cdot |u_m - u_l|$$



Let $u^{\delta}(x,t)$ be defined on some interval $(0,T)$, $T>0$.

$$L(t)=\sum|\nu|$$

Let I be the collision times in $(0,T)$. $L(t)$ is piecewise constant on $(0,T)$. $L(t)$ is well-defined on $t\epsilon(0,T)\setminus I$.

$$\Delta L(t)=L(t+)-L(t-) \qquad t\,\epsilon\, I\ .$$

$\forall\, t\,\epsilon(0,T)\setminus I$,

$$Q(t)=\sum{}' |\alpha|\cdot|\beta|$$

(α,β) are approaching waves acrossing t-time line.

α is i-wave, β is j-wave, either $i>j$ or $i=j$ and one of them must be a compressive shock.

$Q(t)$ is also piecewise constant, and
$\Delta Q(t) = Q(t+) - Q(t-) \leq 0 \quad \forall\, t \,\epsilon\, I$

$$\begin{aligned} \Delta L(t) &= O(1)|\alpha| \cdot |\beta| \qquad t \,\epsilon\, I \\ Q(t) &= \sum{}' |\alpha| \cdot |\beta| \\ Q(t+) - Q(t-) &= -|\alpha| \cdot |\beta| + O(1)|\alpha| \cdot |\beta| \cdot L(t-) \end{aligned}$$

choose a constant k,

$$\begin{aligned} G(t) &= L(t) + k\, Q(t) \\ \Delta G(t) &= \Delta L(t) + k\, \Delta Q(t) \qquad t \,\epsilon\, I \end{aligned}$$

$t \epsilon I$,

$$\begin{aligned} \Delta G(t) &\leq O(1)|\alpha|\cdot|\beta| + k(-1 + O(1)\cdot L(t-))|\alpha|\cdot|\beta| \\ &= (O(1) - k(1 - O(1)\cdot L(t-)))|\alpha|\cdot|\beta| \end{aligned}$$

If $O(1)\, L(t-) \leq \frac{1}{2}$, $\;k \geq 4O(1)$, then

$$\Delta G(t) \leq -\frac{k}{4}|\alpha|\cdot|\beta|$$

Claim: By induction, $\Delta G(t) \leq -\frac{k}{4}|\alpha| \cdot |\beta|, \quad \forall\, t \,\epsilon\, I$, if $L(0+)O(1) \leq \eta_0$

$$
\begin{aligned}
G(t_1+) &\leq G(t_1-) = G(0+) \\
&= L(0+) + k\,Q(0+) \\
&\leq L(0+) + \frac{k}{2} L^2(0+) \\
&\leq 2L(0+) \\
L(t_1+) &\leq 2L(0+)\ .
\end{aligned}
$$

By induction, we prove that $L(t) \leq \eta_1$,
$\Delta G(t) \leq -\frac{k}{4}|\alpha| \cdot |\beta| \quad \forall\, \epsilon\, \epsilon\, I$.

Step 3 Estimates on total interactions.

$\forall\, t \in I$,

$$
\begin{aligned}
\Delta Q(t) &= (-1 + O(1)\cdot L(t-))|\alpha|\cdot|\beta| \le -\frac{1}{2}|\alpha|\cdot|\beta| \\
\frac{1}{2}{\sum}'|\alpha|\cdot|\beta| &\le -\sum \Delta Q(t) = Q(0+) - Q(T) \\
&\le Q(0+) \le \frac{1}{2}L^2(0+)
\end{aligned}
$$

Step 4 Estimates on the number of collision.

The key is to estimate the number of collisions which have to be resolved by ARS, which will be used only when the incoming interacting waves satisfy $|\alpha| \cdot |\beta| \geq \sigma$.

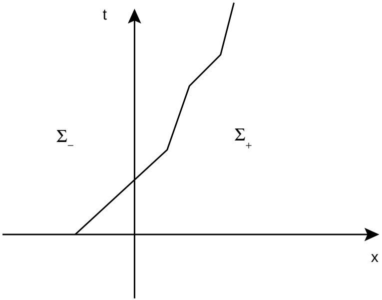
$$\begin{aligned} \sum_{t \in I}{}' |\alpha| \cdot |\beta| &= \sum_{|\alpha||\beta| \geq \sigma}{}' |\alpha| \cdot |\beta| + \sum_{|\alpha| \cdot |\beta| < \sigma}{}' |\alpha| \cdot |\beta| \\ N\sigma &\leq \sum_{|\alpha||\beta| \geq \sigma}{}' |\alpha| \cdot |\beta| \leq \frac{1}{2} L^2(0+) \\ N &\leq \frac{1}{2\sigma} L^2(0+) \end{aligned}$$

$\Rightarrow$ total number of collisions is finite (e.x.).
$\Rightarrow$ $u^\delta(x,t)$ can be defined on $(0, +\infty)$.

Step 5 Total variation estimates on a non-resonant curve.

Definition 3.10 A Lipschitz continuous curve is said to be non-resonant if it divides the half plane into positive $\sum^{+}$ and negative $\sum^{-}$.
Further $\{1, \cdots, n, n+1\}$ can be decomposed into N^{+}, N^{0}, N^{-} such that:

1. N^{0} contains at most one point,
2. N^{+}, N^{0}, N^{-} are pairwise disjoint,
3. N^{+} and N^{-} contain consecutive numbers in $\{1, 2, \cdots, n, n+1\}$,

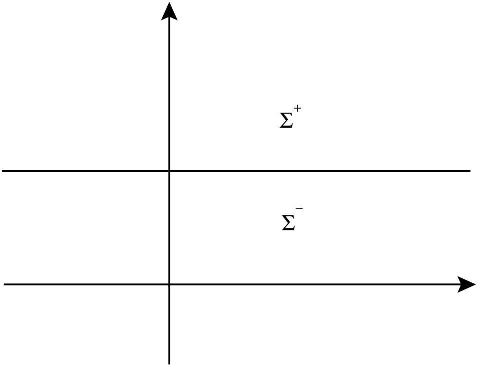


If $i \in N^-$, and i-characteristic hits c, then it crosses c from $\sum^+$ to $\sum^-$.
If $i \in N^+$, and i-characteristic hits c, then it crosses c from $\sum^-$ to $\sum^+$.
If $i \in N^0$, and i-characteristic hits c, then it must become part of c (it can hit c from both sides).

(Here i-characteristic means i-approximate characteristic.)

Example 1: $t =$ constant, $c > 0$ is non-resonant, $N^0 = \phi$, $N^- = \phi$,

$N^+ = \{1, \cdots, n, n+1\}$.



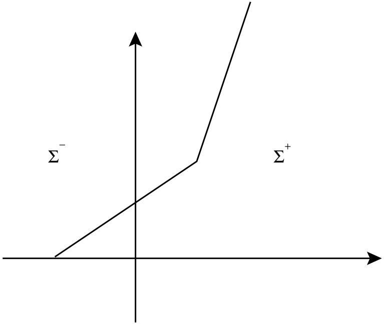
Example 2: Any space-like curve. Assuming $\lambda_1(u) < 0 < \lambda_{n+1}(u)$, these may be represented by Lipschitz functions $t = \hat{t}(x)$, $-\infty < x < \infty$, with $\frac{1}{\lambda_1} < \frac{d\hat{t}_1}{dx} < \frac{1}{\lambda_{n+1}}$, a.e. on $(-\infty, +\infty)$.

In that case $\mathcal{N}_+ = \{1, \cdots, n+1\}$ while $\mathcal{N}_- = \mathcal{N}_0 = \phi$.

Example 3: *cisi*-characteristics.

X_i: ith-characteristics

$$\begin{aligned} N^0 &= \{i\} \\ N^- &= \{1, 2, \cdots, i-1\} \\ N^+ &= \{i+1, \cdots, n+1\} \end{aligned}$$



Let c be Lipsehitz and non-resonant with respect to u^δ .
$T\,V\,u^\delta|_c = \sum |\nu|$, ν are all the waves cross c .

Let J be the times that some waves hits on c ,

$$M(t) = \sum_{-} |\nu| + \sum_{+} |\nu| + \sum_{0} |\nu| \qquad t \,\epsilon (0, T) \setminus (I\ U\ J)$$

$\sum_-$: sums over all the i-wave, $i \,\epsilon\, N^-$, cross t-time line on the positive side.

$\sum_+$: sums over all the i-wave, $i \,\epsilon\, N^+$, cross t-time line on the negative side.

$\sum_0$: sums over all the i-wave, with $i \,\epsilon\, \mathcal{N}^0$, which cross t-time line on either side of e .

$$\begin{array}{rcll} \Delta M(t) & = & -\nu & t \, \epsilon \, J \setminus I \\ \Delta M(t) & = & O(1)|\alpha||\beta| & t \, \epsilon \, I \setminus J \\ \Delta M(t) & = & -|\nu| + O(1)|\alpha||\beta| & t \, \epsilon \, I \cap J \end{array}$$

where $|\alpha|$ and $|\beta|$ are the strengths of the waves colliding at $t \, \epsilon \, I$ and $|\nu|$ is the strength of the wave that impinges on c at $t \, \epsilon \, J$.

$$T\,V\,u^{\delta}|_{c} \leq M(T) \leq M(0) + k \sum{}' |\alpha||\beta| \leq 2L(0+)$$

Step 6 Estimate the total strength of the pseudo shocks.

Main idea: Waves of higher generation order are produced after a large number of collisions. So it should be expected to be small if its initial strength is small.

Step 6.1 Estimate on the total strength of waves of higher generation order.

Since the total number of collisions of waves is finite, the generation order is finite also, $\exists \nu > 0$, $0 \leq \mu \leq \nu$. However, $\nu = \nu(\delta)$ and in general, $\nu(\delta) \to +\infty$ if $\delta \to 0$.

Definition 3.11

(1.) $L_\mu(t) = \sum |\nu|$, $|\nu|$ across t-time line and $\mu(\nu) \geq \mu$.

(2.) $Q_\mu(t) = \sum' |\alpha| \cdot |\beta|$, where α and β are approaching waves.

Both cross t-time line, moreover, $\max\{\mu(\alpha), \mu(\beta)\} \geq \mu$.

(3.) $I_\mu = \{t \epsilon I$; at which a wave of order μ collides with a wave of order $\leq \mu\}$.

In particular, $L_0(t) = L(t)$, $Q_0(t) = Q(t)$. How to estimate $L_\mu(t)$ and $Q_\mu(t)$ when μ large?

Lemma 3.8

(1.)	$\Delta L_\mu(t) = 0$	$t \epsilon I_0 \, U \, I_1 \, U \cdots U I_{\mu-2}$
(2.)	$\Delta L_\mu(t) + 2k\,\Delta Q_{\mu-1}(t) \leq 0$	$t \epsilon I_{\mu-1} \, U \cdots U I_\nu$
(3.)	$\Delta Q_\mu(t) + 2k\,\Delta Q(t)\,L_\mu(t-) \leq 0$	$t \epsilon I_0 \, U \cdots U I_{\mu-2}$
(4.)	$\Delta Q_\mu(t) + 2k\,\Delta Q_{\mu-1}(t)\,L(t-) \leq 0$	$t \epsilon I_{\mu-1}$
(5.)	$\Delta Q_\mu(t) \leq 0$	$t \epsilon I_\mu \, U \cdots U I_\nu$

Proof of Lemma 3.8

(1.) $\Delta L_\mu(t) = 0 \qquad t \,\epsilon\, I_1 \, U \cdots U I_{\mu-2}$

This follows, interactions among waves of generation order $\leq \mu - 2$, can produce waves of the generation order $\leq \mu - 1$, which has no effects on $L_\mu(t)$.

(2.) can be proved similarly. It follows from this lemma that

Claim: If η is small, then

$$\begin{aligned} \hat{L}_\mu &= \sup_t L_\mu(t) &&\leq 2^{-\mu}\, c\, \eta \\ \hat{Q}_\mu &= \sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)]^+ &&\leq 2^{-\mu+3}\, c^2\, \eta^2 \quad (T.V.\, u_0 \leq \eta_0) \end{aligned}$$

$[A]^+ = \max\{A, 0\}$, $[A]^- = \max\{-A, 0\}$

Proof of the claim:

Part 1: Note that $L_\mu(0+) = 0$, $\mu = 1, 2, \cdots, \nu$, then it follows (1.) + (2.) (in fact, sum them up) to get

$$\begin{aligned} L_\mu(t) &\le \sum_{t \,\epsilon\, I_{\mu-1} U \cdots U I_\nu} (-2k\,\Delta Q_{\mu-1}(t)) \\ &\le 2k \sum_{t \,\epsilon\, I} [\Delta Q_{\mu-1}(t)]^- \\ \text{so} \qquad \hat{L}_\mu &\le 2k \sum_{t \,\epsilon\, I} [\Delta Q_{\mu-1}(t)]^- \end{aligned}$$

Part 2: Estimate on the potential $Q_\mu(0+) = 0, \quad \mu = 1, 2, \cdots, \nu,$

$$\begin{aligned} L(t) &= L_0(t) \le G(t) \le G(0+) \\ &\le L(0+) + \frac{1}{2}L^2(0+) \\ &\le 2L(0+) \end{aligned}$$

$$\begin{aligned} \sum [\Delta Q(t)]^- &= Q(0+) - Q(T-) \le Q(0+) \\ &\le \frac{1}{2}L^2(0+) \end{aligned}$$

(3.) + (4.) + (5.)

$$\begin{aligned}\sum_{t\,\epsilon\,I}[\Delta Q_\mu(t)]^+ &\le 2k\sum L_\mu(t-)\,[\Delta Q(t)]^- + 2k\sum[\Delta Q_{\mu-1}]^- L(t-)\\ &\le 2k\,\hat{L}_\mu\sum_{t\,\epsilon\,I}[\Delta Q(t)]^- + 4k\,L(0+)\sum_{t\,\epsilon\,I}[\Delta Q_{\mu-1}]^-\\ &\le 2k\,\hat{L}_\mu\,\frac{1}{2}L^2(0+) + 4k\,L(0+)\sum[\Delta Q_{\mu-1}]^-\end{aligned}$$

Therefore,

$$\begin{aligned}
(2) \quad \hat{Q}_\mu &\leq 2k \cdot 2k \sum [\Delta Q_{\mu-1}(t)]^- \cdot \frac{1}{2} L^2(0+) \\
&\quad + 4k\, L(0) \sum [\Delta Q_{\mu-1}(t)]^- \\
\text{so} \quad \hat{Q}_\mu &\leq \frac{1}{2} \sum_{t \,\epsilon\, I} [\Delta Q_{\mu-1}(t)]^-, \quad \mu = 1, 2, \cdots, \nu,
\end{aligned}$$

when $L(0+)$ is small enough.

In particular, for $\mu = 1$,

$$\hat{Q}_1 \leq \frac{1}{2} \sum [\Delta Q]^- \leq \frac{1}{4} L^2(0+)$$

However,

$$\sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)]^- = \sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)]^+ - \sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)], \qquad \mu = 1, 2, \cdots$$

since $[A]^+ - [A]^- = A$, $Q_\mu(0+) = 0$,

$$\begin{array}{lccccc} \text{hence} & \sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)]^- & \leq & \sum_{t \,\epsilon\, I} [\Delta Q_\mu(t)]^+ & \equiv & \hat{Q}_\mu \\ \text{so} & \hat{Q}_\mu & \leq & \frac{1}{2}\hat{Q}_{\mu-1} & & \mu = 1, 2, \cdots, \nu. \end{array}$$

This implies the claim by induction.

Step 6.2 Estimate of the combined strength of pseudo shocks of generation order $\leq \mu_0$.

Part 1: Bound on the number of waves of generation order $\leq \mu_0$. Let k_μ be the number of waves of order $\leq \mu$. Then
k: wave strength

$$\begin{aligned} k_\mu &\leq k_{\mu-1} + \frac{1}{2}(k_{\mu-1}^2)\frac{k}{\delta}n \\ &\leq \frac{b}{\delta}k_{\mu-1}^2 \\ \Longrightarrow \quad k_\mu &\leq \left(\frac{b}{\delta}\right)^{2^{\mu+1}} c^{2^\mu}, \end{aligned}$$

c depends on initial data.

Part 2: Estimate on the strength of a pseudo shock. Let α be the strength of the pseudo shock.

Claim: $|\alpha| \leq c\,\sigma$

Proof of the claim: This is true initially. The strength of the wave will change only when it interacts with front of strength β, by the estimate, now the strength will be

$$O(1)\sigma(1 + |\beta|)$$

as the strength at any time is bounded by

$$\begin{aligned} & O(1)\sigma \cdot \Pi(1 + |\beta|) \leq O(1)\sigma\, e^{\sum |\beta|} \\ = & O(1)\sigma \cdot e^{O(1) - L(0+)} = O(1)\sigma \end{aligned}$$

Conclusion: The combined strength of pseudo shocks of order $\mu_0 \leq k_{\mu_0}\, \sigma\, O(1) < \frac{\varepsilon}{2}$ (choose σ small enough).

$$\forall\, \varepsilon > 0, \quad \exists\, \mu_0 \text{ such that} \quad \hat{L}_{\mu_0} < 2^{-\mu_0}\, c\, \eta < \frac{\varepsilon}{2}$$

then fix μ_0, choose σ so small that

$$k_{\mu_0}\, O(1)\sigma < \frac{\varepsilon}{2}$$

$\Longrightarrow$ the combined strength of all pseudo shocks of any order $\leq \varepsilon$.
Consequently $\Longrightarrow$ Theorem 3.6

Theorem 3.7 Let u_δ be a sequence of δ-approximate solution constructed by the front-tracking algorithm. Then $\exists$ subsequence $\delta_k \to 0$ as $k \to \infty$, such that

$$u_{\delta_k} \to u, \qquad \text{a.e.} \quad \mathbb{R}^1 \times (0, \infty)$$

such that

(i) $u(\cdot, t) \epsilon BV$, which is a weak solution to (1.1)-(1.2).

(ii) u satisfies the entropy admissible condition.

(iii) $T.V.\, u(\cdot, t) \leq c\, T.V.\, u_0 \quad 0 \leq t < +\infty$.

(iv) $\int_{-\infty}^{\infty} |u(x,t) - u(x,s)| dx \leq c\, |t - s|\, T.V.\, u_0$.

(v) the trace of u on any Lipschitz continuous graph in $\mathbb{R}' \times \mathbb{R}'_+$ which is non-resonant to u_δ has bounded total variation.

Proof:

Step 1: Recall that the δ-approximate solution u_δ constructed by our front tracking algorithm satisfies

$$\begin{array}{rcll} T.V.\, u_\delta(\cdot, t) & \leq & c\, T.V.\, u_0 & \forall t \geq 0 \\ \displaystyle\int_\infty^\infty |u_\delta(x,t) - u_\delta(x,s)| dx & \leq & c\, T.V.\, u_0 (t-s) & \forall t \geq s \ (e.x.) \end{array}$$

The same arguments using Helley's principle and diagonal procedure

$$\Longrightarrow \exists\, \delta_k \to 0 \quad \text{as} \quad k \to +\infty, \quad \text{s.t.} \quad u_\delta \to u \quad \text{a.e.} \quad \mathbb{R}' \times \mathbb{R}'_+$$

Step 2: Consistency of the front tracking method

$$\begin{array}{ll} \text{Aim :} & \partial_t u + \partial_x f(u) = 0 \quad \text{in } \mathcal{D} \\ \text{i.e.} & \forall \, \varphi \, \epsilon \, c_0^\infty(\mathbb{R}^{'} \times \mathbb{R}^{'}_{+}) \end{array}$$

$$\int\int \partial_t \varphi \, u(x,t) + \partial_x \varphi \, f(u(x,t)) dx \, dt + \int_{-\infty}^{\infty} \varphi(x,0) \, u_0(x) dx = 0$$

Therefore, define

$$\begin{array}{lll} E_\delta(\varphi) & = & \int\int \{\partial_t \varphi \, u_\delta + \partial_x \varphi \, f(u_\delta)\} dx \, dt + \int \varphi(x,0) \, u_\delta(x,0) dx \\ \text{so} & & E_\delta \to 0 \quad \text{as} \quad \delta \to 0+ \\ \text{iff} & & \text{the scheme is consistent.} \end{array}$$

Recall that $u_\delta(x,t)$ is a piecewise constant function with FINITELY many jumps which are called $x = x_\alpha(t)$ $(\alpha < N)$. By using Green's theorem and direct computation, we have,

$$E_\delta = -\int_0^{+\infty} \sum_\alpha \varphi([f(u_\delta)] - \dot{x}_\alpha[u_\delta])(x_\alpha(t), t)dt$$

where $[A] = A(x_\alpha+) - A(x_\alpha-)$. The summation runs over all the jumps of u_δ at the time t.

Case 1: $x = x_\alpha(t)$ is an approximate shock or contact discontinuity.

Claim: $|([f(u_\delta)] - \dot{x}_\alpha(t)[u_\delta])(x_\alpha(t), t)| \leq \delta |[u_\delta](x_\alpha(t), t)|$

Proof of the claim: Recall the shock speed $s = \lambda_i(u_\delta^+, u_\delta^-)$ by R-H condition

$$[f(u_\delta)] - s[u_\delta] = 0$$

so

$$\begin{aligned} |[f(u_\delta)] - \dot{x}_\alpha[u_\delta]| &\leq |[f(u_\delta)] - s[u_\delta]| + |s[u_\delta] - \dot{x}_\alpha[\dot{u}_\delta]| \\ &\leq 0 + \delta|[u_\delta]| \\ &= \delta|[u_\delta]| \end{aligned}$$

Case 2: $x = x_\alpha(t)$ is an approximation rarefaction front.

Recall that the shock wave curve and the rarefaction wave curve are at least 2nd order contact.

$$
\begin{aligned}
\left|\left([f(u_\delta)] - \dot{x}_\alpha[u_\delta]\right)\right| &\le |[f(u_\delta)] - s[u_\delta]| + |s[u_\delta] - \dot{x}_\alpha[u_\delta]| \\
&= 0 + |s - \dot{x}_\alpha| \, |[u_\delta]| \\
&\le c\,\delta |[u_\delta]|
\end{aligned}
$$

Case 3: $x = x_\alpha(t)$ is a pseudo shock.

$$|[f(u_\delta)] - \dot{x}_\alpha[u_\delta]| \leq c\,|[u_\delta]|$$

By cases 1-3, we have

$$\begin{aligned}
|E_\delta| &\leq c\,\varphi\left(\sum_{\alpha\,\epsilon\,p}|[f(u_\delta)] - \dot{x}_\alpha[u_\delta]| + \sum_{\alpha\,\epsilon\,\mathcal{N}\,p}|[f(u_\delta)] - \dot{x}_\alpha[u_\delta]|\right)\\
&\leq c\,\varphi(c\,\delta\,T.V\,u_0 + c\,\delta)\\
&\leq c\,\varphi\,c(1 + T.V\,u_0)\delta\\
&\longrightarrow 0 \qquad \text{as} \quad \delta \to 0+
\end{aligned}$$

Step 3: Entropy solution

Let $(\eta(u), q(u))$ be an entropy-entropy flux with η convex. Let $\varphi \, \epsilon \, c_c^\infty(\mathbb{R}^{'} \times \mathbb{R}^{'}_{+})$, with $\varphi \geq 0$. Then applying Green's theorem, we can get

$$\begin{aligned} & \iint \{\partial_t \varphi \; \eta(u_\delta) + \partial_x \varphi \; q(u_\delta)\} dx \, dt + \int \varphi(x, t=0) \; u_\delta^0 \, dx \\ = \; & -\int_0^{+\infty} \sum_\alpha \varphi([q(u_\delta)] - \dot{x}_\alpha(t)[\eta(u_\delta)]) dt \end{aligned}$$

Case 1: $x = x_\alpha(t)$ is i-entropy shock

$$\begin{aligned} & [q(u_\delta)](x_\alpha(t), t) - \dot{x}_\alpha(t)[\eta(u_\delta)](x_\alpha(t), t) \\ = \; & [q(u_\delta)] - s[\eta(u_\delta)] + (s - \dot{x}_\alpha(t))[\eta(u_\delta)] \\ \leq \; & (s - \dot{x}_\alpha(t))[\eta(u_\delta)] \\ \leq \; & \delta |[\eta(u_\delta)]| \\ \leq \; & c\,\delta |[u_\delta]| \end{aligned}$$

Case 2 & 3: Similar as before. So we have

$$
\begin{aligned}
& -\int_0^{+\infty} \sum_\alpha \varphi([q(u_\delta)] - \dot{x}_\alpha(t)[\eta(u_\delta)])dt \\
\geq \quad & -c\,\varphi\,\delta(T.V.\,u_0 + 1) \to 0 \qquad \text{as} \quad \delta \to 0
\end{aligned}
$$

Step 4: Proof of (5.)

An easy self exercise.

§3.8 Continuous Dependence of the front Tracking Solutions

Recall (scalar case)

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & u \epsilon \mathbb{R}' \\ u(x, t = 0) = u_0(x) \end{cases}$$

<u>L^1-contraction principle</u>

Let u and v be two "right" solutions to (1) with initial data u_0 and v_0 respectively. Then

$$\int_{-\infty}^{\infty} |u(x,t) - v(x,t)| dx \leq \int_{-\infty}^{\infty} |u_0(x) - v_0(x)| dx$$

$\exists$ example due to B temple. NO L^1-contraction in $n \times n$ system.
Our aim:

$$\int_{-\infty}^{\infty} |u(x,t) - v(x,t)| dx \leq c \int_{-\infty}^{\infty} |u_0(x) - v_0(x)| dx$$

Bressan's idea: In a non-translation invariant space,

$$\frac{1}{c} ||u - v||_{L^1} \leq p(u,v) \leq c ||u - v||_{L^1}$$

$$p(u,v)(t) \leq p(u,v)(s) \qquad \forall\, t \geq s$$

Let u and υ be δ-approximate solutions to (1.1). For fixed t, then

$$\upsilon(x) = s^n_{p_n(x)} \circ \cdots \circ s^1_{p_1(x)} u(x)$$

Define

$$p(u(x), \upsilon(x)) = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} w_i(x) \, |p_i(x)| \, dx$$

where $w_i(x)$ are weights to be determined.

If $1 \leq w_i(x) \leq 2$, then

$$\int_{-\infty}^{\infty} \sum |p_i(x)| \, dx \leq p(u, \upsilon) \leq 2 \int_{-\infty}^{\infty} \sum |p_i(x)| \, dx$$

$$\frac{1}{c} \int_{-\infty}^{\infty} |u(x) - \upsilon(x)| \, dx \leq p(u, \upsilon) \leq c \int_{-\infty}^{\infty} |u(x) - \upsilon(x)| \, dx$$

The crucial part is how to define $w_i(x)$

$$w_i(x) = 1 + k_1 \, A_i(x) + k_2(Q(u) + Q(\upsilon))$$

where $Q(u(t))$ is the potential of wave interaction amount approaching waves acrossing time t. $A_i(x)$ are the total strength of physical wave in u and υ which approach the i-wave $p_i(x)$.

$$A_i(x,t) = \begin{cases} \left[\sum_{\substack{x_\alpha < x \\ i < k_\alpha \le n}} + \sum_{\substack{x_\alpha > x \\ 1 \le i < k_\alpha}} \right] |\sigma_\alpha| & \text{if} \quad i \; Ldg \\ \left[\sum_{\substack{x_\alpha < x \\ i < k_\alpha \le n}} + \sum_{\substack{x_\alpha > x \\ 1 \le i < k_\alpha}} \right] |\sigma_\alpha| + \begin{cases} \left[\sum_{\substack{k_\alpha = i \\ \alpha \in J(u), x_\alpha < x}} + \sum_{\substack{k_\alpha = i \\ \alpha \in J(v), x_\alpha > x}} \right] |\sigma_\alpha| \text{ if } p_i(x) < 0 \\ \left[\sum_{\substack{k_\alpha = i \\ \alpha \in J(u), x_\alpha > x}} + \sum_{\substack{k_\alpha = i \\ \alpha \in J(v), x > x_\alpha}} \right] |\sigma_\alpha| \text{ if } p_i(x) > 0 \end{cases} & \text{if } i - gNL \end{cases}$$

Theorem 3.8 $\exists$ suitable constants δ_0, k_1, k_2, $c > 0$, s.t. if u and v are δ-approximate solution constructed by front tracking algorithm with $G(u(t)) \leq \delta_0$, $G(v(t)) \leq \delta_0$. Then

$$p(u(t), v(t)) - p(u(s), v(s)) \leq c\,\delta(t - s) \quad \forall\, 0 \leq s\, c\, t \quad \forall\, 0 \leq s \leq t$$

Proof: The key is to understand the evolution of p in time,

$$p(u, v) = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} w_i(x)\ |p_i(x)|\ dx$$

Step 1: (At collosion time)

$$t = \tau \, \epsilon \, I \; U \; I^{'}$$

where I: collosion time of u, $I^{'}$: collosion time of υ.

First, note that $p_i(x,t) : [0,+\infty) \longrightarrow L^1(\mathbb{R}^1)$ are continuous at $t = \tau$.

Next, let σ and σ^1 be fronts in u which collide at $t = \tau$, then

$$\begin{array}{rcl} Q(u(\tau+)) - Q(u(\tau-)) & \leq & -\frac{1}{2}|\sigma|\,|\sigma^1| \\ A_i(x,\tau+) - A_i(x,\tau-) & = & O(1)\,|\sigma|\,|\sigma^1| \end{array}$$

Recall

$$w_i = 1 + k_1\, A_i + k_2(Q(u) + Q(\upsilon))$$

so

$$w_i(x, \tau+) - w_i(x, \tau-) \leq 0,$$

if k_2 is large enough.

Therefore

$$p(u, \upsilon)(\tau+) \leq p(u, \upsilon)(\tau-)$$

Step 2: Let u and υ be two δ-approximate solutions.
Fixed (x, t)

$$u(x,t) = s^n_{p_2} \circ \cdots s^2_{p_2} \circ s^1_{p_1}(\upsilon(x,t))$$

$\forall\, t \in\; I_u \cup I_\upsilon$.

In this case, $\frac{d}{dt}p(u,\upsilon)(t)$ is continuously differentiable. To compute $\frac{d}{dt}p(u,\upsilon)(t)$, we set $w_0(x), \cdots, w_n(x)$ by

$$w_0(x) = u(x), \cdots, w_n(x) = \upsilon(x) \quad \text{by}$$

$$\begin{array}{rcl} w_i(x) & = & s^i_{p_i(x)} \circ s^{i-1}_{p_{i-1}(x)} \circ \cdots \circ s^1_{p_1(x)}(w_0) \\ s_i & = & \lambda_i(w_{i-1}(x), w_i(x)) \end{array}$$

Let $x_1(t), x_2(t), \cdots, x_\alpha(t), x_N(t)$ be all the point where either u or v has a jump.

Claim:

$$\begin{aligned} & \frac{d}{dt} p(u, v)(t) \\ = & \sum_{\alpha \epsilon J} \sum_{i=1}^{n} \dot{x}_\alpha [|p_i| \, w_i](x_\alpha) \\ = & \sum_{\alpha \epsilon J} \sum_{i=1}^{n} \dot{x}_\alpha \{ |p_i(x_\alpha+)| \; w_i(x_\alpha+) - |p_i(x_\alpha-)| \; w_i(x_\alpha-) \} \end{aligned} \tag{3.33}$$

Proof of the claim:

$$p(u,v)=\int_{-\infty}^{x_1(t)}\sum_{i=1}^{n} w_i(x,t)\,|p_i(x,t)|\,dx+\sum_{\alpha=1}^{N-1}\int_{x_\alpha(t)}^{x_{\alpha+1}}+\int_{x_N(t)}^{+\infty}$$

To estimate the right hand side of (3.33), we will denote

$$p_i^{\alpha+}=p_i(x_\alpha+),\ p_i^{\alpha-}=p_i(x_\alpha-),\ w_i^{\alpha\pm}=w_i(x_\alpha\pm),\ \lambda_i^{\alpha\pm}=s_i(x_\alpha\pm)$$

Since u and v are constants on $(x_{\alpha-1}(t),x_\alpha(t))$, then

$$|p_i(x)|\ \lambda_i(x)\ w_i(x)=|p_i^{(\alpha-1)+}|\ \lambda_i^{(\alpha-1)+}\ w_i^{(\alpha-1)+}=|p_i^{\alpha-}|\ \lambda_i^{\alpha-}\ w_i^{\alpha-}$$

then

$$
\begin{aligned}
& \frac{d}{dt}\, p(u, \upsilon)(t) \\
= \; & \sum_{\alpha \,\epsilon\, J} \sum_{i=1}^{n} \{ |p_i^{\alpha +}| \; w_i^{\alpha +}(\lambda_i^{\alpha +} - \dot{x}_\alpha) - |p_i^{\alpha -}| \; w_i^{\alpha -}(\lambda_i^{\alpha -} - \dot{x}_\alpha) \} \\
= \; & \sum_{\alpha \,\epsilon\, J} \sum_{i=1}^{n} E_{\alpha,i}(t)
\end{aligned}
$$

where

$$
E_{\alpha,i} = |p_i^{\alpha +}| \; w_i^{\alpha +}(\lambda_i^{\alpha +} - \dot{x}_\alpha) - |p_i^{\alpha -}| \; w_i^{\alpha -}(\lambda_i^{\alpha -} - \dot{x}_\alpha)
$$

Proposition 3.3

$$\sum_{i=1}^{n} E_{\alpha\, i} \leq O(1)\ |\sigma_\alpha| \qquad \alpha \,\epsilon\, \mathcal{N}\, p \tag{3.34}$$

$$\sum_{i=1}^{n} E_{\alpha\, i} \leq O(1)\,\delta\,|\sigma_\alpha| \qquad \alpha \,\epsilon\, s \,\cap\, R \tag{3.35}$$

Proof: Let us start with $\alpha \,\epsilon\, \mathcal{N}\, p$.

$$
\begin{aligned}
E_{\alpha,i} &= w_i^{\alpha+}\,|p_i^{\alpha+}|\,(\lambda_i^{\alpha+} - \dot{x}_\alpha) - w_i^{\alpha-}\,|p_i^{\alpha-}|\,(\lambda_i^{\alpha-} - \dot{x}_\alpha) \\
&= (w_i^{\alpha+} - w_i^{\alpha-})\,(\lambda_i^{\alpha+} - \dot{x}_\alpha)\,|p_i^{\alpha-}| \\
&\quad + w_i^{\alpha-}(\lambda_i^{\alpha+} - \lambda_i^{\alpha-})\,|p_i^{\alpha-}| \\
&\quad + w_i^{\alpha+}(\lambda_i^{\alpha+} - \dot{x}_\alpha)\,(|p_i^{\alpha+}| - |p_i^{\alpha-}|) \\
w_i^{\alpha+} - w_i^{\alpha-} &= k_1(A_i^{\alpha+} - A_i^{\alpha-}) \;=\; k_1|\sigma_\alpha| \\
\lambda_i^{\alpha+} - \lambda_i^{\alpha-} &= O(1)\,|\sigma_\alpha| \\
|p_i^{\alpha+}| - |p_i^{\alpha-}| &= O(1)\,|\sigma_\alpha|
\end{aligned}
$$

Key thing is to show (3.35).

$$\text{Fix } \alpha \qquad w_i^{\pm} = w_i^{\alpha\pm}, \quad p_k^{\pm} = p_k^{\alpha\pm}, \quad \upsilon^{\pm} = \upsilon(x_\alpha \pm)$$

Proof of (3.35):

Step 1: Reduction to a single shock case. x_α is a jump point of υ. So that $\alpha \epsilon J(u)$, we introduce the

$$\hat{\upsilon}(x) = s_{\sigma_\alpha}^{k_\alpha}(\upsilon_-), \quad \dot{\hat{x}}_\alpha = \lambda_{k_\alpha}(\upsilon_-, \hat{\upsilon})$$

Define $\hat{p}_i(x)$ such that $\hat{\upsilon} = s_{\hat{p}_n}^n \circ \cdots s_{\hat{p}_1}^1(u)$, the intermediate state $\hat{w}_0 = u(x)$, $\hat{w}_1, \cdots, \hat{w}_n = \hat{v}$, $\hat{w}_i = s_{\hat{p}_i}^i \hat{w}_{i-1}$, $\hat{\lambda}_i = \lambda_i(\hat{w}_{i-1}, \hat{w}_i)$.

Case 1: x_α is a shock or a contact discontinuity, by

$$\begin{gathered}\hat{\upsilon} = \upsilon_+, \quad \hat{w}_i = w_i^+, \quad \hat{\lambda}_i = \lambda_i^+, \quad \forall\, i \\ \hat{p}_i = p_i^+ \\ |\hat{\dot{x}}_\alpha - \dot{x}_\alpha| < \delta \end{gathered} \tag{3.36}$$

Case 2: x_α is rarefaction front, so $0 < \sigma_\alpha \leq \delta$. In this case, since the shock wave curve and rarefaction wave curve are second order contact, so

$$\begin{aligned} \hat{\upsilon} &= \upsilon_+ + O(1)\,|\sigma_\alpha|^3 \\ \hat{p}_i &= p_i^+ + O(1)\,|\sigma_\alpha|^3 \\ \hat{w}_i &= w_i^+ + O(1)\,|\sigma_\alpha|^3 \\ \hat{\lambda}_i &= \lambda_i^+ + O(1)\,|\sigma_\alpha|^3 \\ |\dot{x}_\alpha - \hat{x}_\alpha| &= O(1)\,\delta \end{aligned} \tag{3.37}$$

$$
\begin{aligned}
E_{\alpha,i} &= w_i^{+}\ |p_i^{+}|(\lambda_i^{+} - \dot{x}_\alpha) - w_i^{-}\ |p_i^{-}|\ (\lambda_i^{-} - \dot{x}_\alpha) \\
&= w_i^{\alpha+}\ |p_i^{+}|(\lambda_i^{+} - \hat{\dot{x}}_\alpha) - w_i^{-}\ |p_i^{-}|\ (\lambda_i^{-} - \hat{\dot{x}}_\alpha) \\
&\quad + (\hat{\dot{x}}_\alpha - \dot{x}_\alpha)(w_i^{+}\ |p_i^{+}| - w_i^{-}\ |p_i^{-}|) \\
&= \{w_i^{+}\ |\hat{p}_i|(\hat{\lambda}_i - \hat{\dot{x}}_\alpha) - w_i^{-}\ |p_i^{-}|\ (\lambda_i^{-} - \hat{\dot{x}}_\alpha)\} \\
&\quad + \{w_i^{+}\ |\hat{p}_i|\ (\lambda_i^{+} - \hat{\lambda}_i) + w_i^{+}\ (|p_i^{+}| - |\hat{p}_i|)(\lambda_i^{+} - \hat{\dot{x}}_\alpha)\} \\
&\quad + (\hat{\dot{x}}_\alpha - \dot{x}_\alpha)(w_i^{+}|p_i^{+}| - w_i^{-}|p_i^{-}|) \\
&\equiv E^1_{\alpha_i} + E^2_{\alpha_i} + E^3_{\alpha_i}
\end{aligned}
$$

Claim: $\forall\, i \,\epsilon \{1, 2, \cdots, n\}$

$$E^2_{\alpha,i} = \begin{cases} 0 & \text{if} \quad \sigma_\alpha < 0 \\ O(1)\ |\sigma_\alpha|^3 & \text{if} \quad \sigma_\alpha \,\epsilon [0, \delta] \end{cases} \tag{3.38}$$

$$E^3_{\alpha,i} = O(1)\, \delta\, |\sigma_\alpha| \tag{3.39}$$

Proof of (3.38): $\sigma_\alpha < 0$, x_α is a shock or contact discontinuity, then (3.36) is true. Then

$$\begin{aligned} E^2_{\alpha,i} &= w_i^+\ |\hat{p}_i|\ (\lambda_i^+ - \hat{\lambda}_i) + w_i\ (|p_i^+| - |\hat{p}_i|)\ (\lambda_i^+ - \dot{\hat{x}}_\alpha) \\ &= 0 \end{aligned}$$

if $\quad \sigma_\alpha \,\epsilon [0, \delta]$,

$$E^2_{\alpha,i} = O(1)\ |\sigma_\alpha|^3$$

Proof of (3.39):

$$E^3_{\alpha,i} = (\hat{\dot{x}}_\alpha - \dot{x}_\alpha)\{w_i^+ \,(|p_i^+| - |p_i^-|) + (w_i^+ - w_i^-)|p_i^-|\}$$

By construction, $w_i^+ - w_i^- = k_1(A_1^+ - A_1^-) = O(1)\,k_1\,|\sigma_\alpha|$.

Next,

$$\begin{aligned} \big||p_i^+| - |p_i^-|\big| &\leq |p_i^+ - p_i^-| \leq O(1)\,|\sigma_\alpha| \\ E^3_{\alpha,i} &\leq O(1)\,\delta\,|\sigma_\alpha| \end{aligned}$$

It follows that one needs to show that

$$\sum_{i=1}^{n} E^1_{\alpha,i} \leq O(1)\,\delta\,|\sigma_\alpha|$$

Step 2: Some elementary estimates

Proposition 3.4

(1) If the k_α-family is linearly degenerate. Then

$$|\hat{p}_{k_\alpha} - p^-_{k_\alpha} - \sigma_\alpha| + \sum_{i \neq k_\alpha} |\hat{p}_i - p^-_i| = O(1) \sum_{i \neq k_\alpha} |p^-_i| \cdot |\sigma_\alpha| \quad (3.40)$$

(2) If the k_α-family is genuinely nonlinear

$$\begin{aligned} & |\hat{p}_{k_\alpha} - p^-_{k_\alpha} - \sigma_\alpha| + \textstyle\sum_{i \neq k_\alpha} |\hat{p}_i - p^-_i| \\ = \; & O(1)(|p^-_{k_\alpha}||p^-_{k_\alpha} + \sigma_\alpha| + \textstyle\sum_{i \neq k_\alpha} |p^-_i|)|\sigma_\alpha| \end{aligned} \quad (3.41)$$

(3) If the k_α-family is genuinely nonlinear

$$
\begin{aligned}
\hat{\dot{x}}_\alpha - \lambda^-_{k_\alpha} &= \frac{p^-_{k_\alpha} + \sigma_\alpha}{2} + O(1)\left(|p^-_{k_\alpha} + \sigma_\alpha|(|p^-_{k_\alpha}| + |\sigma_\alpha|) + \sum_{i \neq k_\alpha} |p^-_i| \right) \\
\hat{\dot{x}}_\alpha - \hat{\lambda}_{k_\alpha} &= \frac{p^-_{k_\alpha}}{2} + O(1)\left(|p^-_{k_\alpha}|(|p^-_{k_\alpha}| + |\sigma_\alpha|) + \sum_{i \neq k_\alpha} |p^-_i| \right)
\end{aligned}
$$

Lemma 3.9 Let $\Psi(\tilde{p}, p^*, \sigma) \epsilon C^{2,\infty}(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^1)$ with properties:

(a) $\Psi(\tilde{p}, p^*, 0) = \Psi(0, s, -s) = \Psi(0, 0, \sigma) = 0$,
then $\Psi(\tilde{p}, p^*, \sigma) = O(1)(|\tilde{p}| + |p^*||p^* + \sigma|)|\sigma|$

(b) If $\Psi(\tilde{p}, p^*, 0) = 0 = \Psi(0, p^*, \sigma)$,
then $\Psi(\tilde{p}, p^*, \sigma) = O(1)|\tilde{p}| \cdot |\sigma|$

Lemma 3.10 Let $\Psi(\tilde{p}, p^*, \sigma) \epsilon C^{1,\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1)$. Then,

(a) If $$\frac{\partial \Psi}{\partial p^*}(0,0,0) = \frac{\partial \Psi}{\partial \sigma}(0,0,0) = \frac{1}{2}, \quad \Psi(0, s, -s) = 0,$$
then $$\Psi(\tilde{p}, p^*, \sigma) = \frac{p^* + \sigma}{2} + O(1)(|\tilde{p}| + |p^* + \sigma|(|p^*| + |\sigma|)).$$

(b) If $$\frac{\partial \Psi}{\partial p^*}(0,0,0) = \frac{1}{2}, \Psi(0,0,\sigma) = 0, \qquad \forall \sigma,$$
then $$\Psi(\tilde{p}, p^*, \sigma) = \frac{p^*}{2} + O(1)(|\tilde{p}| + |p^*|(|p^*| + |\sigma|)).$$

Proof of Proposition 3.4:

We fix $u_- = u(x_\alpha)$. Then all quantities, υ^-, $\hat{\upsilon}$, λ_i^-, $\hat{p}_i$, $\hat{\lambda}_i$, w_i, $\hat{w}_i$, can be regarded as functions of
$\tilde{p} = (p_1^-, \cdots, p_{k_\alpha-1}^-, p_{k_\alpha+1}^-, \cdots, p_n^-), p^* = p_{k_\alpha}^-, \sigma = \sigma_\alpha$.

$$\Psi(p_1^-, \cdots, p_n^-, \sigma_\alpha) = \Psi(\tilde{p}, p^*, \sigma)$$

Case 1: The k_α-family is linearly degenerate, then set

$$\begin{array}{lrcll} & \Psi_i & = & \hat{p}_i - p_i^- & i \neq k_\alpha \\ & \Psi_{k_\alpha} & = & \hat{p}_{k_\alpha} - p_{k_\alpha}^- - \sigma_\alpha & \\ i \neq k_\alpha & \Psi_i(\tilde{p}, p^*, 0) & = & 0 & \\ & \Psi_i(0, p^*, \sigma) & = & 0 & \end{array}$$

$S_\sigma^{k_\alpha} \circ S_{p_\alpha^-}^{k_\alpha}\ u_- = S_{\sigma + p_\alpha^-}^{k_\alpha}\ u_-$ depends on that the k_α-family is linearly degenerate.

Case 2: If k_α-family is genuinely nonlinear,

$$\hat{\upsilon} = S_{-s}^{k_\alpha} \circ S_s^{k_\alpha} \; u_- = u_-$$

Case 3:

$$\Psi'_\alpha(\tilde{p}, p^*, \sigma) = \hat{\dot{x}}_\alpha - \lambda_{k_\alpha}^-$$
$$\Psi''_\alpha(\tilde{p}, p^*, \sigma) = \hat{\dot{x}}_\alpha - \hat{\lambda}_{k_\alpha}$$

Step 3: Linearly degenerate fields

Assume that the k_α-family is linearly degenerate $\hat{\upsilon} = \upsilon^+$, $\hat{p}_i = p_i^+$, $\hat{w}_i = w_i^+$, $\hat{\lambda}_i = \lambda_i^+$, $|\hat{\dot{x}}_\alpha - x_\alpha| < \delta$. Then

$$W_{k_\alpha}^+ = W_{k_\alpha}^-$$

Case 1: If $i \neq k_\alpha$, i-th family is linearly degenerate.

$$A_i^+ = \left[\sum_{\substack{x_\beta < x_\alpha^+ \\ i < k_\beta \leq n}} + \sum_{\substack{x_\beta > x_\alpha^+ \\ 1 \leq k_\beta < i}} \right] |\sigma_\beta|$$

$$A_i^- = \left[\sum_{\substack{x_\beta < x_\alpha^- \\ i < k_\beta \leq n}} + \sum_{\substack{x_\beta > x_\alpha^- \\ 1 \leq k_\beta < i}} \right] |\sigma_\beta|$$

$$\begin{array}{llcl} i < k_\alpha & A_i^+ - A_i^- & = & |\sigma_\alpha| \\ i > k_\alpha & A_i^+ - A_i^- & = & -|\sigma_\alpha| \end{array}$$

In summary,

$$A_i^+ - A_i^- = -\operatorname{sgn}(i - k_\alpha)\ |\sigma_\alpha|$$

so,

$$W_i^+ - W_i^- = -k_1 \operatorname{sgn}(i - k_\alpha)\ |\sigma_\alpha|$$

Case 2: $i \neq k_\alpha$, i-th family is genuinely nonlinear

$$W_i^+ - W_i^- = -k_1 \operatorname{sgn}(i - k_\alpha)\, |\sigma_\alpha|$$
$$\lambda_{k_\alpha}^- - \hat{\dot{x}}_\alpha = \lambda_{k_\alpha}(w_\alpha^-) - \lambda_{k_\alpha}(\upsilon_-) = O(1) \sum_{i > k_\alpha} |p_i^-|$$
$$\hat{p}_{k_\alpha} = p_{k_\alpha}^+ = p_{k_\alpha}^- + \sigma_\alpha + O(1) \sum_{i \neq k_\alpha} |\sigma_\alpha| \cdot |p_\alpha^-|$$

First,

$$
\begin{aligned}
E^1_{\alpha,k_\alpha} &= W^+_{k_\alpha}|\hat{p}_{k_\alpha}|\,(\lambda^+_{k_\alpha} - \hat{\dot{x}}_\alpha) - W^-_{k_\alpha}|p^-_{k_\alpha}|\,(\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha) \\
&\leq W^+_{k_\alpha}|\hat{p}_{k_\alpha}|\,|\lambda^-_{k_\alpha} - \hat{\lambda}_{k_\alpha}| + W^+_{k_\alpha}(|\hat{p}_{k_\alpha}| - |p^-_{k_\alpha}|)|\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha| \\
&\leq W^+_{k_\alpha}|\hat{p}_{k_\alpha}|\,|(\lambda^-_{k_\alpha} - \hat{\lambda}_{k_\alpha})| + O(1)\cdot|\sigma_\alpha|\sum_{i\neq k_\alpha}|p^-_i|
\end{aligned}
$$

$$
\Psi = \Psi(\tilde{p}, p^*, \sigma) = \lambda^-_{k_\alpha} - \hat{\lambda}_{k_\alpha}
$$
$$
\Psi(\tilde{p}, p^*, 0) = 0 = \Psi(0, p^*, \sigma) = 0
$$

By Lemma 3.9, $|\lambda^-_{k_\alpha} - \hat{\lambda}_{k_\alpha}| = O(1)(\sum_{i \neq k_\alpha} |p^-_i|)\ |\sigma_\alpha|$

$$E^1_{\alpha,k_\alpha} \leq O(1)|\sigma_\alpha|(\sum_{i\neq k_\alpha} |p^-_i|)$$

for $i \neq k_\alpha$,

$$\begin{aligned}
E^1_{\alpha,i} &= W^+_i|\hat{p}_i|(\lambda^+_i - \hat{\dot{x}}_\alpha) - W^-_i|p^-_i|(\lambda^-_i - \hat{\dot{x}}_\alpha) \\
&= W^+_i|\hat{p}_i|(\lambda^+_i - \hat{\dot{x}}_\alpha) - k_1\ \mathrm{sgn}(i - k_\alpha)|\sigma_\alpha|\ |p^-_i|(\lambda^-_i - \hat{\dot{x}}_\alpha) \\
&\quad - W^+_i|p^-_i|(\lambda^-_i - \hat{\dot{x}}_\alpha) \\
&= -k_1|\lambda^-_i - \hat{\dot{x}}_\alpha|\cdot|\sigma_\alpha|\cdot|p^-_i| + W^+_i\left\{|\hat{p}_i|(\lambda^+_i - \hat{\dot{x}}_\alpha) - |p^-_i|\ (\lambda^-_i - \hat{\dot{x}}_\alpha)\right\} \\
&\leq -k_1\, c_1|\sigma_\alpha|\ |p^-_i| + W^+_i(|\hat{p}_i| - |p^-_i|)(\lambda^+_i - \hat{\dot{x}}_\alpha) + |p^-_i|(\lambda^-_i - \lambda^+_i)w^+_i \\
&\leq -k_1\, c_1|\sigma_\alpha|\ |p^-_i| + W^+_i|\hat{p}_i - p^-_i|\ |\lambda^+_i - \hat{\dot{x}}_\alpha| + O(1)|p^-_i|\ |\sigma_\alpha|
\end{aligned}$$

so, $i \neq k_\alpha$,

$$E^1_{\alpha,i} \leq -c_1\, k_1 |\sigma_\alpha|\, |p_i^-| + O(1)\Big(\sum_{i \neq k_\alpha} |p_i^-|\Big)\, |\sigma_\alpha|$$

$$\sum_{i=1}^{n} E^1_{\alpha,i} \leq 0$$

if k_1 is big enough.

Step 4: k_α-family is genuinely nonlinear

Step 4.1: Estimate of $E^1_{\alpha,i}$, $i \neq k_\alpha$

Then it follows from definition that

$$W_i^+ = W_i^- - k_1|\sigma_\alpha| \operatorname{sgn}(i - k_\alpha).$$

Consequently,

$$
\begin{aligned}
E^1_{\alpha,i} &= -k_1|\sigma_\alpha|\,|p_i^-|\,|\lambda_i^- - \hat{\hat{x}}_\alpha| + W_i^+\left\{|\hat{p}_i|(\hat{\lambda}_i - \hat{\hat{x}}_\alpha) - |p_i^-|(\lambda_i^- - \hat{\hat{x}}_\alpha)\right\} \\
&\leq -k_1\, c|p_i^-|\,|\sigma_\alpha| + W_i^+\left\{(|\hat{p}_i| - |p_i^-|)(\hat{\lambda}_i - \hat{\hat{x}}_\alpha) + |p_i^-|(\lambda_i^- - \hat{\lambda}_i)\right\} \\
&\leq -c\, k_1|\sigma_\alpha|\,|p_i^-| + W_i^+\left\{|\hat{p}_i - p_i^-|\,|\hat{\lambda}_i - \hat{\hat{x}}_\alpha| + |p_i^-|\,|\lambda_i^- - \hat{\lambda}_i|\right\} \\
&\leq -c\, k_1|\sigma_\alpha|\,|p_i^-| + W_i^+\left\{|\hat{p}_i - p_i^-|\,|\hat{\lambda}_i - \hat{\hat{x}}_\alpha| + |p_i^-|(|\sigma_\alpha|^3 + |\lambda_i^- - \lambda_i^+|)\right\} \\
&\leq -c\, k_1|\sigma_\alpha|\,|p_i^-| + O(1)\left\{\delta|\sigma_\alpha| + |p_{k_\alpha}^-|\,|p_{k_\alpha}^- + \sigma_\alpha| + \sum_{i\neq k_\alpha}|p_i^-|\right\}|\sigma_\alpha| \\
\therefore (\bigstar_1)\quad E^1_{\alpha,i} &\leq -c\, k_1|\sigma_\alpha|\,|p_i^-| + O(1)\left\{\delta|\sigma_\alpha| + |p_{k_\alpha}^-|\,|p_{k_\alpha}^- + \sigma_\alpha| + \sum_{i\neq k_\alpha}|p_i^-|\right\}|\sigma_\alpha|
\end{aligned}
$$

Step 4.2: Estimate of E^1_{α,k_α}.

Case 1: $|\sigma_\alpha| \le \delta$, $|p^-_{k_\alpha}| \le 2|\sigma_\alpha|$.

Then (3.4) $\Rightarrow$

$$\begin{array}{rcl} |\hat{p}_{k_\alpha} - p^-_{k_\alpha}| & \le & O(1)|\sigma_\alpha| \\ |\hat{\lambda}_{k_\alpha} - \lambda^-_{k_\alpha}| & \le & |\hat{\lambda}_{k_\alpha} - \lambda^+_{k_\alpha}| + |\lambda^+_{k_\alpha} - \lambda^-_{k_\alpha}| \ = \ O(1)|\sigma_\alpha| \\ |\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha| & \le & |\hat{\dot{x}}_{k_\alpha} - \lambda^-_{k_\alpha}| + |\hat{\lambda}_{k_\alpha} - \lambda^-_{k_\alpha}| \end{array}$$

Proposition 3.4 (3) $\Rightarrow$

$$\begin{aligned}
&\le O(1)|\sigma_\alpha| + \frac{|p_{k_\alpha}^- + \sigma_\alpha|}{2} + O(1)\left(|p_{k_\alpha}^- + \sigma_\alpha|\,\left(|p_{k_\alpha}^-| + |\sigma_\alpha|\right) + \sum_{i\neq k_\alpha} |p_i^-|\right) \\
&= O(1)\left(|\sigma_\alpha| + \sum_{i\neq k_\alpha} |p_i^-|\right) \\
&= O(1)\left(\delta + \sum_{i\neq k_\alpha} |p_i^-|\right)
\end{aligned}$$

It follows from $(\bigstar_1)$ and $(\bigstar_2)$ that

$$\begin{aligned}\sum_{i=1}^{n} E^1_{\alpha,i} &= -c\, k_1 |\sigma_\alpha| \sum_{i\neq k_\alpha} |p_i^-| + O(1)\left(\delta + \sum_{i\neq k_\alpha} |p_i|\right)|\sigma_\alpha| \\ &\leq O(1)\delta|\sigma_\alpha| \qquad \text{if } k_1 \text{ is big enough!}\end{aligned}$$

here we have used $(\bigstar_1)$ and the assumption that

$$|\sigma_\alpha| \leq \delta, \qquad |p^-_{k_\alpha}| \leq 2|\sigma_\alpha| \leq 2\delta.$$

Case 2: $p^-_{k_\alpha}$, $p^+_{k_\alpha}$ and $\hat{p}_{k_\alpha}$ all have the same signs, say all > 0.

Recall

$$A^{\pm}_{k_\alpha} = A_{k_\alpha}(x^{\pm}_\alpha) = \left[\sum_{\substack{\beta \epsilon J \\ x_\beta < x^{\pm}_\alpha, k_\alpha < k_\beta \leq n}} + \sum_{\substack{\beta \epsilon J \\ x_\beta > x^{\pm}_\alpha, k_\alpha > k_\beta \geq 1}} \right] |\sigma_\beta|$$

$$+ \begin{cases} \left[\displaystyle\sum_{\substack{k_\beta = k_\alpha \\ \beta \epsilon J(u), x_\beta < x^{\pm}_\alpha}} + \sum_{\substack{k_\beta = k_\alpha \\ \beta \epsilon J(v), x_\beta > x^{\pm}_\alpha}} \right] |\sigma_\beta| & \text{if } p_{k_\alpha}(x^{\pm}_\alpha) < 0 \\ \left[\displaystyle\sum_{\substack{k_\beta = k_\alpha \\ \beta \epsilon J(v), x_\beta < x^{\pm}_\alpha}} + \sum_{\substack{k_\beta = k_\alpha \\ \beta \epsilon J(u), x_\beta > x^{\pm}_\alpha}} \right] |\sigma_\beta| & \text{if } p_{k_\alpha}(x^{\pm}_\alpha) \geq 0 \end{cases}$$

Since

$$\begin{array}{lll} & p_k^{\pm} > 0, & A_{k_\alpha}^{+} - A_{k_\alpha}^{-} = |\sigma_\alpha| \\ (\text{if} & p_k^{\pm} < 0, & A_{k_\alpha}^{+} - A_{k_\alpha}^{-} = -|\sigma_\alpha|) \\ & \Rightarrow & W_{k_\alpha}^{+} = W_{k_\alpha}^{-} + k_1|\sigma_\alpha| \operatorname{sgn}(p_{k_\alpha}^{-}) \end{array}$$

Next, set

$$\psi(\tilde{p}, p^*, \sigma_\alpha) = \hat{p}_{k_\alpha}(\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha) - p_{k_\alpha}^{-}(\lambda_{k_\alpha}^{-} - \hat{\dot{x}}_\alpha)$$

then

$$\begin{array}{ll} \psi(\hat{p}, p^*, 0) = 0 & (\because \hat{v} = v_-, \hat{p}_i = p_i^-, \hat{\dot{\lambda}}_{k_\alpha} = \lambda_{k_\alpha}^-) \\ \psi(0, 0, \sigma_\alpha) = \hat{p}_{k_\alpha}(\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha) = 0 & \end{array}$$

Since $\tilde{p} = 0,\ \ p^* = p^-_{k_\alpha} = 0,\ \ v_- = u_-,$

$$\begin{aligned} \therefore \quad \hat{v} &= S^{k_\alpha}_{\sigma_\alpha}(v_-) = S^{k_\alpha}_{\sigma_\alpha}(u_-) \\ &\Rightarrow \hat{p}_{k_\alpha} = \sigma_\alpha,\ \hat{p}_i = 0,\ i \neq k_\alpha \end{aligned}$$

$$\begin{aligned} \hat{x}_\alpha &= \lambda_{k_\alpha}(v_-, \hat{v}) = \lambda_{k_\alpha}(u_-, \hat{v}) \\ &\Rightarrow \hat{\lambda}_{k_\alpha} = \lambda_{k_\alpha}(\hat{W}_{k_{\alpha-1}}, \hat{W}_{k_\alpha}) = \lambda_{k_\alpha}(u_-, \hat{v}) \end{aligned}$$

$$\therefore \quad \hat{\dot{x}}_\alpha = \hat{\lambda}_{k_\alpha}$$

Next, we compute $\psi(0, s, -s)$.

Since $\tilde{p} = 0$, $p_i^- = 0$, $p_{k_\alpha}^- = s$, $\sigma_\alpha = -s$, $\hat{v} = S_{-s}^{k_\alpha}(v_-)$, $v_- = S_s^{k_\alpha}(u_-)$,

$$\therefore \quad \begin{array}{l} v_- = S_{+s}^{k_\alpha}(u_-),\ \hat{v} = S_{-s}^{k_\alpha}(v_-) = S_{-s}^{k_\alpha} \circ (s_{+s}^{k_\alpha}(u_-)) = u_- \\ \hat{p}_i = 0,\ \hat{\dot{x}}_\alpha = \lambda_{k_\alpha}(v_-, \hat{v}) = \lambda_k(v^-, u^-) \end{array}$$

On the other hand,

$$\begin{array}{rcl} \lambda_{k_\alpha}^- & = & \lambda_{k_\alpha}(W_{k_{\alpha-1}}, W_{k_\alpha}) \\ & = & \lambda_{k_\alpha}(u^-, v^-) \end{array}$$

$$\therefore \psi(0, s, -s) = -s(\lambda_{k_\alpha}^- - \hat{\dot{x}}_\alpha) \ = \ -s(\lambda_{k_\alpha}(u^-, v^-) - \lambda_{k_\alpha}(u^-, v^-)) \ = \ 0$$

By the Lemma 3.9,

$$\psi(\tilde{p}, p^*, \sigma_\alpha) = O(1)\left(\sum_{i \neq k_\alpha} |p_i^-| + |p_{k_\alpha}^-|\ |p_{k_\alpha}^- + \sigma_\alpha|\right)|\sigma_\alpha|$$

Consequently, one gets

$$\begin{aligned}
E^1_{\alpha,k_\alpha} &= W^+_{k_\alpha}|\hat{p}_{k_\alpha}|\ (\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha) - W^-_{k_\alpha}|p^-_{k_\alpha}|\ (\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha) \\
&= (W^+_{k_\alpha} - W^-_{k_\alpha})\ |p^-_{k_\alpha}|\ (\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha) \\
&\quad + W^+_{k_\alpha}\left\{|\hat{p}_{k_\alpha}|\ (\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha) - |p^-_{k_\alpha}|\ (\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha)\right\} \\
&= -k_1|\sigma_\alpha|\ \mathrm{sgn}\ p^-_{k_\alpha}|p^-_{k_\alpha}|\ \frac{p^-_{k_\alpha} + \sigma_\alpha}{2} \\
&\quad + O\left(|p^-_{k_\alpha} + \sigma_\alpha|\ |p^-_{k_\alpha}| + \textstyle\sum_{i \neq k_\alpha} |p^-_i|\right)|\sigma_\alpha|
\end{aligned}$$

By (3.41) and $\hat{p}_{k_\alpha}$ has same sign as $p^-_{k_\alpha}$, also $|\sigma_\alpha|$ is small:

$$\leq -\frac{k_1}{2}|\sigma_\alpha|\,|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha|+O(1)\left(|p^-_{k_\alpha}+\sigma_\alpha|\;(|p^-_{k_\alpha}|)+\sum_{i\neq k_\alpha}|p^-_i|\right)|\sigma_\alpha|$$

$$\begin{aligned}\therefore (\bigstar_3)\; E^1_{\alpha,k_\alpha} \quad \leq \quad & -\frac{k_1}{2}|\sigma_\alpha|\,|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha| \\ & +O(1)\left(|p^-_{k_\alpha}+\sigma_\alpha|\;(|p^-_{k_\alpha}|)+\sum_{i\neq k_\alpha}|p^-_i|\right)|\sigma_\alpha|\end{aligned}$$

It follows from $(\bigstar_1)$ and $(\bigstar_3)$,

$$
\begin{aligned}
\sum_{i=1}^{n} E^1_{\alpha,i} \quad \leq \quad & -\frac{k_1}{2}|\sigma_\alpha|\ |p^-_{k_\alpha}|\ |p^-_{k_\alpha}+\sigma_\alpha| - c\, k_1|\sigma_\alpha| \sum_{i\neq k_\alpha} |p^-_i| \\
& + O(1)|\sigma_\alpha|\ |p^-_{k_\alpha}|\ |p^-_{k_\alpha}+\sigma_\alpha| \\
& + O(1) \sum_{i\neq k_\alpha} (|p^-_i|)|\sigma_\alpha| + O(1)\ \delta\ |\sigma_\alpha| \\
\leq \quad & O(1)\ \delta\ |\sigma_\alpha| \qquad\qquad \text{if} \quad k_1 \text{ and } k_2 >> 1
\end{aligned}
$$

<u>Case 3:</u> $p^+_{k_\alpha} < 0 < p^-_{k_\alpha}$.

In this case, we may assume that $\sigma_\alpha < 0$, i.e., the front is a shock, otherwise, by (16), one may get into a case like Case 1. Then,

$$\hat{p}_i = p_i^+,\ \hat{w}_i = w_i^+,\ \hat{\lambda}_i = \lambda_i^+,\ \text{and } |\dot{\hat{x}}_\alpha - \dot{x}_\alpha| < \delta.$$

It follows from (3.4) that

$$\begin{aligned} |(p^-_{k_\alpha} - p^+_{k_\alpha}) + \sigma_\alpha| &\leq O(1)\left(|p^-_{k_\alpha}|\,|p^-_{k_\alpha} + \sigma_\alpha| + \sum_{i\neq k_\alpha} |p_i^-|\right)|\sigma_\alpha| \\ &= |(|p^-_{k_\alpha}| + |p^+_{k_\alpha}|) - |\sigma_\alpha|| \\ \therefore \quad \frac{1}{2}|\sigma_\alpha| &\leq |p^-_{k_\alpha}| + |p^+_{k_\alpha}| \ \leq\ 2|\sigma_\alpha| \end{aligned}$$

so,

$$
\begin{aligned}
E^1_{\alpha,k_\alpha} &= W^+_{k_\alpha}\ |\hat{p}_{k_\alpha}|(\hat{\lambda}_{k_\alpha} - \hat{\dot{x}}_\alpha) - W^-_{k_\alpha}|p^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \hat{\dot{x}}_\alpha) \\
&= W^+_{k_\alpha}|\hat{p}_{k_\alpha}|\left(-\frac{p^-_{k_\alpha}}{2} + O(1)|p^-_{k_\alpha}|\ (|p^-_{k_\alpha}| + |\sigma_\alpha|) + O(1)\sum_{i\neq k_\alpha}|p^-_i|\right) \\
&\quad -W^-_{k_\alpha}|p^-_{k_\alpha}|\left(-\frac{p^-_{k_\alpha} + \sigma_\alpha}{2} + O(1)|p^-_{k_\alpha} + \sigma_\alpha|\ (|p^-_{k_\alpha}| + |\sigma_\alpha|)\right. \\
&\quad \left. + O(1)\sum_{i\neq k_\alpha}|p^-_i|\right)
\end{aligned}
$$

Note that,

$$
\begin{aligned}
& W^{+}_{k_\alpha}|\hat{p}_{k_\alpha}|\left(-\frac{p^{-}_{k_\alpha}}{2}\right) - W^{-}_{k_\alpha}|p^{-}_{k_\alpha}|\left(-\frac{p^{-}_{k_\alpha}+\sigma_\alpha}{2}\right) \\
= \; & W^{+}_{k_\alpha}(-|\hat{p}_{k_\alpha}|)\frac{p^{-}_{k_\alpha}}{2} + W^{-}_{k_\alpha}|p^{-}_{k_\alpha}|\frac{p^{-}_{k_\alpha}+\sigma_\alpha-\hat{p}_{k_\alpha}}{2} + W^{-}_{k_\alpha}|p^{-}_{k_\alpha}|\frac{\hat{p}_{k_\alpha}}{2} \\
= \; & \frac{(W^{+}_{k_\alpha}+W^{-}_{k_\alpha})}{2}(-|\hat{p}_{k_\alpha}|)|p^{-}_{k_\alpha}| + W^{-}_{k_\alpha}|p^{-}_{k_\alpha}|\frac{p^{-}_{k_\alpha}+\sigma_\alpha-\hat{p}_{k_\alpha}}{2}
\end{aligned}
$$

On the other hand, (3.4) $\Rightarrow$ ($\because\ \hat{p}_{k_\alpha} = p^+_{k_\alpha}$)

$$|p^-_{k_\alpha} + \sigma_\alpha| - |\hat{p}_{k_\alpha}| \le O(1)\left(|p^-_{k_\alpha}|\ |p^-_{k_\alpha} + \sigma_\alpha| + \sum_{i\neq k_\alpha} |p^-_i|\right)|\sigma_\alpha|$$

$$\therefore\ -|\hat{p}_{k_\alpha}| \le -|p^-_{k_\alpha} + \sigma_\alpha| + O(1)\left(|p^-_{k_\alpha}|\ |p^-_{k_\alpha} + \sigma_\alpha| + \sum_{i\neq k_\alpha} |p^-_i|\right)|\sigma_\alpha|$$

$$\begin{aligned} & w^+_{k_\alpha}|\hat{p}_{k_\alpha}|\left(-\frac{p^-_{k_\alpha}}{2}\right) - w^-_{k_\alpha}|p^-_{k_\alpha}|\left(-\frac{p^-_{k_\alpha} + \sigma_\alpha}{2}\right) \\ \le\ & -|p^-_{k_\alpha}|\ |p^-_{k_\alpha} + \sigma_\alpha| + O(1)|p^-_{k_\alpha}|\left(|p^-_{k_\alpha}|\ |p^-_{k_\alpha} + \sigma_\alpha| + \sum_{i\neq k_\alpha} |p^-_i|\right)|\sigma_\alpha| \end{aligned}$$

$$(\bigstar_4) \ \therefore \ E^1_{\alpha,k_\alpha}$$

$$\begin{aligned}
\leq \quad & -|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha| + O(1)(|p^-_{k_\alpha}+\sigma_\alpha|\,|p^-_{k_\alpha}|\,(|p^-_{k_\alpha}|+|\sigma_\alpha|)) \\
& +O(1)(|\hat{p}_{k_\alpha}|+|p^-_{k_\alpha}|)\sum_{i\neq k_\alpha}|p^-_i| \\
& +O(1)|p^-_{k_\alpha}|\left(|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha|+\sum_{i\neq k_\alpha}|p^-_i|\right)|\sigma_\alpha| \\
\leq \quad & -|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha| + O(1)\left(|p^-_{k_\alpha}|\,|p^-_{k_\alpha}+\sigma_\alpha|+\sum_{i\neq k_\alpha}|p^-_i|\right)|\sigma_\alpha|
\end{aligned}$$

It follows from $(\bigstar_1)$ and $(\bigstar_4)$ that

$$
\begin{aligned}
& \sum_{i=1}^{n} E^1_{\alpha,i} \\
\leq \;& -|p^-_{k_\alpha}|\;|p^-_{k_\alpha}+\sigma_\alpha| - c\,k_1 \left(\sum_{i\neq k_\alpha} |p^-_i|\right)|\sigma_\alpha| \\
& + O(1)\;\delta|\sigma_\alpha|^2 + O(1)\left(|p^-_{k_\alpha}|\;(|p^-_{k_\alpha}+\sigma_\alpha|) + \sum_{i\neq k_\alpha}|p^-_i|\right)|\sigma_\alpha| \\
\leq \;& O(1)\;\delta\;|\sigma_\alpha| \qquad\qquad \text{we are done}
\end{aligned}
$$

Case 4: All other cases:

All the other cases can be reduced to the one of three cases above. So the proof is accomplished.

As a direct consequence of Theorem 3.8, we have the following existence of a semigroup of solutions:

Theorem 3.9 Existence of a Semigroup of solutions

Consider $\mathcal{D} =$ closure $u \in L^1(\mathbb{R}; \mathbb{R}^n)$, u is piecewise constant, $G(u) < \delta_0$.

Then $\exists\ \delta_0 > 0$ with the following property: Let $\bar{u} \in \mathcal{D}$, and u^δ be δ_- approximate solution of the Cauchy problem

$$\begin{cases} \partial_t\, u + \partial_x\, f(u) &= 0 \\ u(x, t = 0) &= \bar{u}(x) \end{cases}$$

Then as $\delta \to 0$, u^δ converges to unique limit solution $u : [0, \infty) \mapsto \mathcal{D}$. The map $(\bar{u}, t) \mapsto u(\cdot, t) = S_t\, \bar{u}$ is a uniformly Lapselitz continuous semigroup. Indeed, $\exists$ constant C and C^1 such that $\forall\, \bar{u}, \bar{v}, \in \mathcal{D}, s, t \geq 0$, one has

$$S_0\, \bar{u} = \bar{u}, \qquad S_s \circ (s_t\, \bar{u}) = S_{s+t}\, \bar{u}$$
$$||s_t\, \bar{u} - s_s\, \bar{v}||_{L^1} \leq c||\bar{u} - \bar{v}||_{L^1} + c^1|t - s|$$

Proof is trivial.