# MATH 5070 Exam 1 Solutions.

1. proof: The tangent bundle of $S^n$ can be written as

$$p: E_1 \longrightarrow S^n$$

where $E_1 = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x|=1 \quad x \perp v\}$

We think of $v$ as a tangent vector to $S^n$ by translating it so that its tail is at the head of $x$ on $S^n$.

Thus
$$p: E_1 \longrightarrow S^n$$
$$(x, v) \longmapsto x$$

To construct local trivializations choose any point $x \in S^n$ and let $U_x \in S^n$ to be one open hemisphere which contains $x$ and bounded by a hyperplane through the origin orthogonal to $x \in \mathbb{R}^n$. Define

$$h_x : p^{-1}(U_x) \longrightarrow U_x \times p^{-1}(x) \cong U_x \times \mathbb{R}^n \quad \text{by } h_x(y, v) = (y, \pi_x(v))$$

where $\pi_x$ is orthogonal projection onto the hyperplane $p^{-1}(x)$. Then $h_x$ is the local trivialization. Since $\pi_x$ is the isomorphism of $p^{-1}(y)$ onto $p^{-1}(x)$ for every $y \in U_x$. So if $(x, v) \in E_1$ we can express it by $(x_0, \cdots x_n, y_0 \cdots y_n) \in E_1$ where

$$\begin{cases} x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \\ x_0 y_0 + x_1 y_1 + \cdots + x_n y_n = 0 \end{cases}$$

$$E = \{(z_0, \cdots z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_n^2 = 1\} = \left\{(a_0, b_0, a_1, b_1 \cdots, a_n, b_n) \in \mathbb{R}^{2n+2} \;\middle|\; \begin{cases} a_0^2 + a_1^2 + \cdots + a_n^2 - b_0^2 \cdots - b_n^2 = 1 \\ a_0 b_0 + \cdots + a_n b_n = 0 \end{cases}\right\}$$

where $z_j = a_j + i b_j \quad (j = 0, 1, 2 \cdots n)$

Thus we construct the function $f: E \longrightarrow E_1$

$$(a_0, b_0, \cdots a_n, b_n) \longmapsto \left(\frac{a_0}{\sqrt{1+b_0^2+\cdots+b_n^2}} \cdots \frac{a_n}{\sqrt{1+b_0^2+\cdots+b_n^2}}, b_0, \cdots b_n\right)$$

This function and its inverse are differentiable since $(0, 0 \cdots 0) \notin E$ and $(0, 0 \cdots 0) \notin E$. Thus $f$ defines a diffeomorphism from $E$ to $TS^n$.

2. proof: According to the definition of manifolds with boundary, we can prove that if $x \in \partial X$, we can find a neighborhood $U_x$ and a diffeomorphism

$$h_x: U_x \longrightarrow D_x$$

Here $D_x$ is an open set in $H^n = \{(x_1 \cdots x_n) \mid x_n \geq 0\} \subseteq \mathbb{R}^n$.

Let $g: H^n \longrightarrow \mathbb{R}$ be the projection. Then the composition $g \circ h_x$ defines
$(x_1 \cdots x_n) \longmapsto x_n$

a function on $U_x$ which is nonnegative and $(g \circ h_x)^{-1}(0) = U_x \cap \partial X$

So $d(g \circ h_x)_z(\vec{n}(z)) > 0$ for $z \in U_x \cap \partial X$ and $\vec{n}$ is the unit normal vector at $z$

As $\partial X$ is paracompact, it has a locally finite cover $\{U_i\}_{i \in \Lambda}$. Each $U_i$ corresponds to a function $h_i$ which is defined above. Then we have $f_i = g \circ h_i \geq 0$ and $f_i(x) = 0$ if and only if $x \in \partial X$. The union of $\{U_i\}_{i \in \Lambda}$ and $X - \partial X$ is a locally finite cover for $X$. By partition of unity, we can find functions $g_i: X \longrightarrow [0,1]$ and $F: X \longrightarrow [0,1]$ such that

$\mathrm{supp}(g_i) \subseteq U_i$, $\mathrm{supp}(F) \subseteq X - \partial X$ and $\sum_{i \in \Lambda} g_i(x) + F(x) = 1$ for $\forall x \in X$.

For $\forall x \in X$, define a function $f = \sum_{i \in \Lambda} g_i f_i + F$ which is nonnegative.

Let $f(x) = 0$, then $g_i(x) f_i(x) = 0$ and $F(x) = 0$. If $x \notin \partial X$, and $x \notin \bigcup_{i \in \Lambda} U_i$, $F(X) = 1$, which is a contradiction. If $x \notin \partial X$ and $x \in \bigcup_{i \in \Lambda} U_i$, $x$ must belong some open set $U_j$, $j \in \Lambda$. $g_j(x) f_j(x) > 0$. It is also a contradiction. Thus $x \in \partial X$

In reverse, $f(x) = 0$ for $\forall x \in \partial X$. Since $F(x) = 0$ and $f_i(x) = 0$ for $\forall x \in \partial X$

So $f(x) = 0$ iff $x \in \partial X$

$$f'(x) = \sum_{i\in\Lambda} g_i'(x) f_i(x) + \sum_{i\in\Lambda} g_i(x) f_i'(x) + F'(x)$$

Let $x \in \partial X$, $y = h_1(x)$. There is a finite number $g_1 \cdots g_k$ in $\{g_i\}_{i\in\Lambda}$ satisfy

$g_1(x) \neq 0$ $g_2(x) \neq 0$ $\cdots$ $g_k(x) \neq 0$. Denote $U_1 \cap U_2 \cap U_3 \cdots \cap U_k = V_x \ni x$. Then

the function $f \circ h_1^{-1}$ is from $V_x$ to $\mathbb{R}$. It suffices to check $(f\circ h_1^{-1})'(0, 0 \cdots 1) \neq 0$

we assume the transition functions are $\varphi_{12}, \varphi_{13} \cdots \varphi_{1k}$. Then

$$(f \circ h_1^{-1})(y) = (g_1 \circ h_1^{-1})(y)\, g(y) + \cdots + (g_k \circ h_1^{-1})(y)\,(g \circ \varphi_{1k})(y)$$

$$(f \circ h_1^{-1})'|_y = (g_1 \circ h_1^{-1})(y) \cdot g'|_y + \cdots + (g_k \circ h_1^{-1})(y) \cdot (g \circ \varphi_{1k})'|_y$$

$g(x_1, \cdots x_n) = x_n$ $g'(0, 0 \cdots 1) = 1$

Since $\varphi_{1i}$ $i = 1, 2 \cdots k$ are diffeomorphisms from $h_1(U_1)$ to $h_i(U_i)$

Then $(g \circ \varphi_{1i})(y) \geq 0$ $g(y) \geq 0$ and $(g \circ \varphi_{1i})'(0, 0 \cdots 1) \geq 0$

Thus $(f \circ h_1^{-1})'(0, 0 \cdots 1) > 0$, and $0$ is a regular value for the function $f$.

3. proof: Since $X$ is a codimensional 1 submanifold, its normal bundle $N$ is an $\mathbb{R}$-bundle over $X$. This is assumed to be nontrivial.

By the tubular neighborhood theorem (cf. e.g. [Lang] p. 111), there is a tubular neighborhood $J(X) \subset M$ diffeomorphism to $N$. $\partial J(X)$ is a $\mathbb{Z}_2$-bundle over $X$. (i.e. it is a double cover of $X$)

Furthermore, this is a nontrivial $\mathbb{Z}_2$-bundle by the nontriviality of the $\mathbb{R}$-bundle $N$.

A nontrivial $\mathbb{Z}_2$-bundle over a connected manifold must be connected: If it is disconnected, then the two disjoint points in any fiber must belong to different connected components, each being a copy of the base manifold, since the latter is connected. (This is because of the unique lifting property of covering space, c.f. e.g. [Bredon], Theorem 4.1)

Now, to show that $M - X$ is connected, it suffices to show that $M - J(X)$ is connected, but the connected manifold $M$ is the union of $J(X)$, $M - J(X)$ along $\overline{J(X)} \cap (M - J(X)) = \partial(M - J(X))$ both $\overline{J(X)}$ and $\partial(M - J(X))$ being connected. Thus $M - J(X)$ must be connected.