MATH6032 ASSIGNMENT 1

DUE FEB 24, 2021

1. Let I be an ideal of a polynomial ring and $\geq$ be a monomial ordering. A Gröbner basis $\{g_1, \ldots, g_n\}$ of I with respect to $\geq$ is called minimal if the coefficients of $in_{\geq}(g_i)$ are all 1 and for any $i \neq j$, $in_{\geq}(g_i) \nmid in_{\geq}(g_j)$.

Show that if $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, l_m\}$ are minimal Gröbner bases. Then $n = m$ and $\{in_{\geq}(g_1), \ldots, in_{\geq}(g_n)\} = \{in_{\geq}(h_1), \ldots, in_{\geq}(h_m)\}$.

2. In the lecture, we have described the procedure to obtain a reduced Gröbner basis. Prove that the reduced Gröbner basis of I with respect to $\geq$ is unique.

3. Let $\geq_1, \geq_2$ be monomial ordering. Prove that if $in_{\geq_1}(I) = in_{\geq_2}(I)$, then the reduced Gröber bases with respect to $\geq_1$ and $\geq_2$ are the same.

4. Let $NP(f)$ be the Newton polytope of a polynoimal f. Prove that $NP(fg) = NP(f) + NP(g)$, where $+$ means the Minkowski sum.

5. Consider a polynomial ring of $2m$ indeterminates $x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}$. Let I be the ideal generated by $D_{ij} = x_{1i}x_{2j} - x_{1j}x_{2j}$. In other words, I is the ideal of 2×2-minors of a $2 \times m$ matrix of indeterminates.

(a) Show that the set $\{D_{ij}\}$ is a universal Gröbner basis of I.

(b) Determine the state polytope and Gröbner fan for I.