Tutorial 8

The heat Kernel on the circle

Heat kernel $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$

$$= (f * H_t)(x).$$

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$ good kernel

(not trivial, in Chap. 5)

Poisson kernel $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}$

$$= (f * P_r)(\theta)$$

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

4.5.11 Show that if $u(x,t) = (f * H_t)(x)$ where H_t is the heat kernel, and f is Riemann integral, then

$$\int_0^1 |u(x,t) - f(x)|^2 dx \to 0 \quad \text{as } t \to 0.$$

Proof 1. $\int_0^1 |u(x,t) - f(x)|^2 dx \le 2\int_0^1 |u(x,t) - S_N(f)(x)|^2 dx$

$$+ 2\int_0^1 |S_N(f) - f(x)|^2 dx.$$

$$\int_0^1 |u(x,t) - S_N(f)(x)|^2 dx \le \int_0^1 \Big|\sum_{|n|\le N} a_n (e^{-4\pi^2 n^2 t} - 1) e^{2\pi i n x}\Big|^2 dx$$

$$+ \int_0^1 \Big|\sum_{|n|>N} a_n e^{-4\pi^2 n^2} e^{2\pi i n x}\Big|^2 dx := I_1 + I_2$$

$$I_1 \le \Big(\sum_{|n|\le N} |a_n|^2\Big)\big(1-e^{-4\pi^2 N^2 t}\big), \quad I_2 \le \sum_{|n|>N} |a_n|^2.$$

~~Therefore for any $\varepsilon>0$, $\exists\, \delta>0$ s.t. for any $t\in(0,\delta)$,~~

$$\cancel{1-e^{-4\pi^2N^2t} < \frac{\varepsilon}{2\|f\|^2}}$$

~~$\exists\, N_0 \in \mathbb{N}^*$ s.t. for any $N>N_0$,~~

$$\cancel{\sum_{|n|>N} |a_n|^2 < \frac{\varepsilon}{4} \text{ and } \int_0^1 |S_N(f)-f|^2 dx < \frac{\varepsilon}{4}}$$

Hence, $\int_0^1 |u(x,t)-f(x)|^2\,dx < \varepsilon$, which implies

$$\int_0^1 |u(x,t)-f(x)|^2\,dx \to 0 \quad \text{as } t\to 0.$$

$\square$

Proof 2.
$$\int_0^1 |u(x,t)-f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |\hat f(n)\hat{H}_t(n)-\hat f(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |\hat f(n)|^2 |\hat H_t(n)-1|^2$$

$$= \sum_{|n|\ge N} |\hat f(n)|^2 |\hat H_t(n)-1|^2 + \sum_{|n|<N} |\hat f(n)|^2 |\hat H_t(n)-1|^2$$

Since for any $\varepsilon>0$, $\exists\, N$ s.t. $\sum_{|n|\ge N} |\hat f(n)|^2 < \varepsilon/2$.

And for that N, $\exists\, \delta$ s.t. for any $0<t<\delta$,

$$|\hat H_t(n)-1|^2 < \frac{\varepsilon}{2\|f\|^2} \quad \text{for any } |n|\le N.$$

$\square$

<u>4.3.17</u> Prove that if f is a bounded monotonic function on $[-\pi, \pi]$, then

$$\hat{f}(n) = O\left(\frac{1}{|n|}\right).$$

(<u>Hint:</u> Prove it is true for the charateristic function, and the sum of the form

$$\sum_{k=1}^{N} \alpha_k \chi_{[a_k, a_{k+1}]}(x)$$

with $-\pi = a_1 < a_2 < \cdots < a_N < a_{N+1} = \pi$, $-M \leq \alpha_1 \leq \cdots \leq \alpha_N \leq M$.)

<u>Proof.</u> For the sum of the form

$$g_N = \sum_{k=1}^{N} \alpha_k \chi_{[a_k, a_{k+1}]}(x)$$

with $-\pi = a_1 < a_2 < \cdots < a_N < a_{N+1} = \pi$, $-M \leq \alpha_1 \leq \cdots \leq \alpha_N \leq M$,

One has

$$\widehat{g_N}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{N} \alpha_k \chi_{[a_k, a_{k+1}]}(x) \right) e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} \alpha_k \left(\int_{a_k}^{a_{k+1}} e^{-inx} \, dx \right)$$

$$= \frac{1}{2\pi} \cdot \frac{1}{-in} \sum_{k=1}^{N} \alpha_k \left(e^{-in a_{k+1}} - e^{-in a_k} \right)$$

$$= \frac{1}{2\pi} \cdot \frac{1}{-in} \left(\alpha_N e^{-in a_{N+1}} - \alpha_1 e^{-in a_1} + \sum_{k=1}^{N-1} (\alpha_k - \alpha_{k+1}) e^{-in a_{k+1}} \right)$$

Therefore, $\left| \widehat{g_N}(n) \right| \leq \frac{1}{2\pi |n|} \left(\sum_{k=1}^{N-1} |\alpha_k - \alpha_{k+1}| + |\alpha_1| + |\alpha_N| \right) \leq \frac{2M}{\pi |n|}$

Obviously, for any montonically increasing function f on $[-\pi,\pi]$, for any $n \in \mathbb{Z}, n \neq 0$, there exists function g_{N_n} in the form above such that

$|f - g_{N_n}| < \frac{1}{|n|}$. Therefore,

$$\left| \hat{f}(n) - \hat{g}_{N_n}(n) \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f - g_{N_n}| dx \leq 1.$$

So $\hat{f}(n) = O\left(\frac{1}{|n|}\right)$. $\square$