Solution to Assignment 8

Ex 3. (p. 162) See Tutorial 10.

Ex 5. (p. 162) (a) For any $h \in \mathbb{R}$,

$$\widehat{f}(\xi + h) - \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\Big(e^{-2\pi i(\xi+h)x} - e^{-2\pi i\xi x}\Big)\, dx.$$

Since f is of moderate decrease, for any $\epsilon > 0$, there exists a $N = N(\epsilon)$ such that $\int_{|x|\geq N} |f|\, dx < \epsilon/4$, and then there exists a $\delta = \dfrac{\epsilon}{4\pi N \int_{-\infty}^{\infty} |f(x)|\, dx}$ such that

$$\begin{aligned}\Big|\widehat{f}(\xi + h) - \widehat{f}(\xi)\Big| &\leq \int_{|x|\leq N} + \int_{|x|\geq N} |f(x)|\Big|e^{-2\pi i(\xi+h)x} - e^{-2\pi i\xi x}\Big|\, dx \\ &< 2\pi h N \int_{-\infty}^{\infty} |f(x)|\, dx + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

This implies that $\widehat{f}$ is continuous.

Note that by changing of variables, one has that

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x - \frac{1}{2\xi})e^{-2\pi i\xi(x-\frac{1}{2\xi})}dx = -\int_{-\infty}^{\infty} f(x - \frac{1}{2\xi})e^{-2\pi i\xi x}\, dx.$$

Therefore,

$$\begin{aligned}|\widehat{f}(\xi)| &= \Big|\frac{1}{2}\int_{-\infty}^{\infty}\Big(f(x) - f(x - \frac{1}{2\xi})\Big)e^{-2\pi i\xi x}\, dx\Big|, \\ &\leq \frac{1}{2}\int_{-\infty}^{\infty}\Big|f(x) - f(x - \frac{1}{2\xi})\Big|\, dx.\end{aligned}$$

Since for moderately decreasing functions, $\int_{-\infty}^{\infty} |f(x+h) - f(x)|dx \to 0$ as $h \to 0$, so does $\widehat{f}(\xi)$ as $\xi \to \pm\infty$.

(b) It is easy to verify the multiplication formula holds whenever $f, g \in \mathcal{M}(\mathbb{R})$ since $F(x,y) := f(x)g(y)e^{-2\pi i xy}$ satisfies $|F(x,y)| \leq \frac{C}{(1+x^2)(1+y^2)}$. Hence, by $\widehat{f}(\xi) \equiv 0$, we have

$$\int_{-\infty}^{\infty} f(x)\widehat{g}(x)\, dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)g(\xi)d\xi = 0 \quad \text{for all} \quad g \in \mathcal{M}(\mathbb{R}).$$

For $x_0 = 0$, take $g(x) = e^{-\pi\delta x^2} \in \mathcal{S}(\mathbb{R})$, we have $\int_{-\infty}^{\infty} f(x)K_\delta(x)dx = 0$, where $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2\delta}$. Since K_δ s a family of good kernel, $f(0) = \lim_{\delta\to 0+}\int_{-\infty}^{\infty} f(x)K_\delta(x)dx = 0$. For any $x_0 \in \mathbb{R}$, just consider $f_{x_0}(x) = f(x - x_0)$ instead of $f(x)$ and repeat the same argument. □

Ex 9. (p. 163) Define $g(\xi) := \chi_{[-R,R]}(\xi)(1 - \frac{|\xi|}{R})$, $f_{-x}(t) := f(t+x)$ where χ is the characteristic function. One can check that (I omit the calculations..): $\widehat{g}(t) = \mathcal{F}_R(t)$, and $\widehat{f_{-x}}(\xi) = \widehat{f}(\xi)e^{2\pi i x\xi}$. Therefore, by the multiplication formula, we have

$$\int_R^R (1 - \frac{|\xi|}{R})\widehat{f}(\xi)e^{2\pi i x\xi} d\xi = \int_{-\infty}^{\infty} g(t)f(t+x)\ dt$$
$$= \int_{\infty}^{-\infty} g(-t)f(-t+x)\ d(-t) = (f * \mathcal{F}_R)(x.)$$

(In the future you may learn if $h_1, h_2 \in \mathcal{M}(\mathbb{R})$ and $\int h_1 e^{2\pi i x\xi} d\xi = h_2(x)$, then $\widehat{h_2} = h_1$. So the above inequality is actually the Fourier inversion for $f * \mathcal{F}$.)

Since $\mathcal{F}_R$ is nonnegative, to prove $\mathcal{F}_R$ is a family of good kernels, it suffices to prove

$$\int_{-\infty}^{\infty} \mathcal{F}_R(t)dt = 1 \quad \text{and} \quad \lim_{R\to+\infty} \int_{|t|\geq\delta} \mathcal{F}_R(t)dt = 0 \quad \text{for any} \quad \delta > 0.$$

Note that $\widehat{g}(t) = \mathcal{F}_R(t)$, $g, \mathcal{F}_R \in \mathcal{M}(\mathbb{R})$, and thus it follows the Fourier inversion that

$$\int_{-\infty}^{\infty} \mathcal{F}_R(t)dt = \int_{-\infty}^{\infty} \mathcal{F}_R(t)e^{2\pi i 0 t}dt = g(0) = 1.$$

Moreover, for any $\delta > 0$,

$$\int_{|t|\geq\delta} \mathcal{F}_R(t)\ dt \leq R\int_{|t|\geq\delta} \frac{1}{\pi R^2 t^2}dt = \frac{2}{\pi R\delta} \to 0 \quad \text{as} \quad R \to \infty.$$

Therefore, $\mathcal{F}_R$ is a family of good kernels. $\square$