Solution to Assignment 6

Ex 4. (p. 122) (a) Suppose that the strong version of isoperimetric inequality holds. Given a 2π-periodic C^1 function $y(s)$, and satisfies $\int_0^{2\pi} y(s)\ ds = 0$, we can define for a curve Γ by the parametrization $\gamma(s) = (x(s), y(s)), s \in [0, 2\pi]$, where $x'(s) = -y(s)$. Then $\int_0^{2\pi} y(s)\ ds = 0$ ensures Γ is a closed curve. Let l be its length and $\mathcal{A}$ be the area of its surrounded part.

It follows the isoperimetric inequality that

$$\begin{aligned} 0 \le 2(\frac{l^2}{4\pi} - \mathcal{A}) &\le \frac{1}{2\pi}\Big(\int_0^{2\pi} \sqrt{x'(s)^2 + y'(s)^2}\ ds\Big)^2 + 2\int_0^{2\pi} y(s)x'(s)\ ds \\ &\le \frac{1}{2\pi}\Big(\int_0^{2\pi} 1\ ds\Big)\Big(\int_0^{2\pi} x'(s)^2 + y'(s)^2\ ds\Big) + 2\int_0^{2\pi} y(s)x'(s)\ ds \\ &= \int_0^{2\pi} [x'(s) + y(s)]^2\ ds + \int_0^{2\pi} (y'(s)^2 - y(s)^2)\ ds \\ &= \int_0^{2\pi} (y'(s)^2 - y(s)^2)\ ds. \end{aligned}$$

Therefore,

$$\int_0^{2\pi} y(s)^2\ ds \le \int_0^{2\pi} y'(s)^2\ ds,$$

where the equality holds if and only if Γ is a circle, i.e. $y = a + r\sin(s + \phi)$ for some $a \in \mathbb{R},\ r > 0,\ \phi \in [0, 2\pi)$. Since $\int_0^{2\pi} y(s)\ ds = 0$, one has $a = 0$. Therefore, $y(s) = A\sin s + B\cos s$, for some constants A, B.

(b)Suppose that Wirtinger's inequality holds. Consider a C^1 curve of length 2π parametrized by $\gamma(s) = (x(s), y(s)),\ s \in [0, 2\pi]$ and $\int_0^{2\pi} y(s)\ ds = 0$, otherwise consider the curve

$$\bar{\gamma}(t) = \Big(\frac{l}{2\pi}x(t), \frac{l}{2\pi}y(t) - \frac{1}{2\pi}\int_0^{2\pi} y(s)ds\Big),$$

where l is the length of γ and change the variable of $\bar{\gamma}$ as arc-length. Then it suffices to prove that if A is the area of the surrounded part,

$$\pi - A \ge 0.$$

Since s is the arc-length variable, $\sqrt{x'(s)^2 + y'(s)^2} = 1$. So

$$\begin{aligned} 2(\pi - \mathcal{A}) &= \int_0^{2\pi} [x'(s) + y(s)]^2\ ds + \int_0^{2\pi} (y'(s)^2 - y(s)^2)\ ds \\ &\ge \int_0^{2\pi} [x'(s) + y(s)]^2\ ds \ge 0, \end{aligned}$$

where in the last row, the Wirtinger's inequality is used. The equality holds if and only if $y(s) = A\sin s + B\cos s$ for some constants A, B and $x'(s) + y(s) = 0$. It is easy to check it is equivalent to that $\gamma(s)$ is a circle. □

Ex 5. (p. 122) See Tutorial 6.

Ex 10a (p.123-124). Since $\{\xi_n\}$ **is equidistributed in [0,1), by the Weyl's criterion,** for $k \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0.$$

Hence,

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(x+\xi_n)}\right| = |e^{2\pi i k x}| \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n}\right| = \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n}\right| \to 0$$

uniformly in x. Hence, by the linearity of the limit, for all trigonometric polynomial $P(x)$ with $\int P(x)dx = 0$,

$$\lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} P(x+\xi_n) = 0$$

uniformly in x. Now for any $\epsilon > 0$ and given any continuous function f with $\int f(x)dx$, we can find trigonometric polynomial P such that $\int P(x)dx = 0$

$$\sup_{x\in[0,1]} |f(x) - P(x)| < \epsilon.$$

Hence,

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right| &\leq \left|\frac{1}{N}\sum_{n=1}^{N}(f(x+\xi_n) - P(x+\xi_n))\right| + \left|\frac{1}{N}\sum_{n=1}^{N} P(x+\xi_n)\right| \\ &\leq \sup_{x\in[0,1]} |f(x) - P(x)| + \left|\frac{1}{N}\sum_{n=1}^{N} P(x+\xi_n)\right| \\ &< \epsilon + \left|\frac{1}{N}\sum_{n=1}^{N} P(x+\xi_n)\right|.\end{aligned}$$

Taking limit in N, we have

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}f(x+\xi_n)\right|<\epsilon.$$

uniformly in x. But ϵ is arbitrary, we have $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x+\xi_n)=0$. This completes the proof.

Ex 10(b). For any $\epsilon>0$ and any Riemann integrable functions f, by Lemma 3.2 in Chapter 2 of the book, there exists a continuous function g such that

$$\sup_{x\in[0,1]}|g(x)|\leq\sup_{x\in[0,1]}|f(x)| \text{ and } \int_0^1|f(x)-g(x)|dx<\epsilon.$$

Define $h(x)=g(x)-\int g(x)dx$. Then h satisfies condition in (a), so that $\frac{1}{N}\sum_{n=1}^{N}h(x+\xi_n)\to 0$ uniformly in x. Hence, this means that for N large

$$\left|\frac{1}{N}\sum_{n=1}^{N}g(x+\xi_n)-\int_0^1 g(x)dx\right|<\epsilon \text{ uniformly in } x.$$

Let $M=\sup_{x\in[0,1]}|f(x)|$, note that $\int f(x)dx=0$, we have

$$\begin{aligned}
\int_0^1\left|\frac{1}{N}\sum_{n=1}^{N}f(x+\xi_n)\right|^2dx\leq& M\int_0^1\left|\frac{1}{N}\sum_{n=1}^{N}f(x+\xi_n)\right|dx\\
\leq& M\int_0^1\left|\frac{1}{N}\sum_{n=1}^{N}(f(x+\xi_n)-g(x+\xi_n))\right|dx\\
&+M\int_0^1\left|\frac{1}{N}\sum_{n=1}^{N}g(x+\xi_n)-\int_0^1 g(x)dx\right|dx\\
&+M\int_0^1\left|\int_0^1 g(x)dx-\int_0^1 f(x)dx\right|dx\\
<& M\int_0^1|(f(x+\xi_n)-g(x+\xi_n)|\,dx+2M\epsilon\\
<&3M\epsilon.
\end{aligned}$$

This establishes the result.