MATH 5061 Riemannian Geometry

Solution to Problem Set 1

Problem 1

We use $[(x,t)]$ to denote the equivalent class of the quotient space M.
(a) We show that M can be covered by two charts $\{(U_i,\phi_i)\}_{i\in\{1,2\}}$.

Choose U_1 be the quotient of $(0,1)\times\mathbb{R}$, $\phi_1 : U_1 \to (0,1)\times\mathbb{R}$ defined by $\phi_1([x,t]) = (x,t)$. Clearly it is well-defined and it is a homeomorphism.

Choose U_2 be the quotient of $((0.5,1]\cup[0,0.5))\times\mathbb{R}$, $\phi_2 : U_2 \to (0.5,1.5)\times\mathbb{R}$ defined by

$$\phi_2([x,t]) := \begin{cases} (x,t), & x > 0.5 \\ (x+1,-t), & x < 0.5 \end{cases}$$

This is well defined since for any $t\in\mathbb{R}$, we have $\phi_2([(0,t)]) = (1,-t) = \phi_2([(1,-t)])$, which says the value of ϕ_2 does not rely on the representative. It's easy to see it is a homeomorphism by the property of quotient space.

So we find two charts covering M.

Moreover, the transaction map $\phi_2\circ\phi_1^{-1} : ((0,0.5)\cup(0.5,1))\times\mathbb{R} \to ((0.5,1)\cup(1,1.5))\times\mathbb{R}$ has the form

$$\phi_2\circ\phi_1^{-1}(x,t) = \begin{cases} (x,t), & x > 0.5 \\ (x+1,-t), & x < 0.5 \end{cases}$$

by the definition of ϕ_i. So it is a C^∞ map and its inverse is still a C^∞ map.

Hence, the charts $\{(U_i,\phi_i)\}_{i\in\{1,2\}}$ define a differentiable structure on M.

(b) Let's assume M is orientable. So there is an atlas $\mathcal{A} = \{(\tilde{U}_i,\tilde{\phi}_i)\}_{i\in I}$ such that all transition maps are orientation-preserving.

Now, we will insert $(U_1,\phi_1),(U_2,\phi_2)$ into the atlas $\mathcal{A}$ to get a contradiction.

First, let's consider a function $f : \phi_1^{-1}(U_1) \to \mathbb{R}$ defined by

$$f(p) := \begin{cases} 1, & \text{if } \phi_1(p)\in\tilde{U}_i \text{ and } \det(d(\tilde{\phi}_i\circ\phi_1)) > 0 \text{ for some } i\in I \\ 0, & \text{if } \phi_2(p)\in\tilde{U}_i \text{ and } \det(d(\tilde{\phi}_i\circ\phi_1)) < 0 \text{ for some } i\in I \end{cases}$$

It is well-defined since if $\phi_1(p)\in\tilde{U}_j$ at the same time, then $\det(d(\tilde{\phi}_j\circ\tilde{\phi}_i^{-1})) > 0$ and $\det(d(\tilde{\phi}_i\circ\phi_1)) > 0 (<0) \implies \det(d(\tilde{\phi}_j\circ\phi_1)) > 0(<0)$.

Note that f is indeed continuous, since for each $\phi_1(p)\in\tilde{U}_i$, we can find small neighborhood V of p with $\phi(V)\subset\tilde{U}_i$ and the function $\det(d(\tilde{\phi}_i\circ\phi_1))$ is also continuous and not vanish everywhere.

Since f can only take two values, we know f is indeed a constant since $\phi_1^{-1}(U_1) = (0,1)\times\mathbb{R}$ is connected. If f always takes 1, then by the definition of f, we know (U_1, ϕ_1) is compatible with $\mathcal{A}$ with the orientation on it. That is we can take $\tilde{\mathcal{A}} = \mathcal{A} \cup \{(U_1, \phi_1)\}$ and $\tilde{\mathcal{A}}$ is an atlas that all transition maps are orientation-preserving.

If f always takes -1, we can reverse the orientation of ϕ_i, i.e. by choosing $\phi_1([x,t]) = (x, -t)$, or reverse the orientation of M to make sure f is greater than 0. For notation simplicity, we just reverse the orientation of M on this case.

Note that by the same trick, we can also add (U_2, ϕ_2) into our orientation chart $\tilde{\mathcal{A}}$ or add $(U_2, \overline{\phi}_2)$ into $\tilde{\mathcal{A}}$ where $\overline{\phi}_2([x,t]) := \phi_2([x,-t])$, which reverses the orientation of ϕ_2.

This shows $\det(d(\phi_2 \circ \phi_1^{-1}))$ should be always positive or negative on $\phi_1^{-1}(U_1 \cap U_2)$.

But we know the exact form of $\phi_2 \circ \phi_1^{-1}$, which imply

$$\det(d(\phi_2 \circ \phi_1^{-1})) = \begin{cases} 1, & x > 0.5 \\ -1, & x < 0.5 \end{cases}$$

This is a contradiction with the above fact.

Hence, M is non-orientable.

(c) We show that $\mathbb{RP}^2 \backslash$a disk is homeomorphic to Möbius band.

Note that $\mathbb{RP}^2$ can be viewed as the quotient space of sphere $\mathbb{S}^2$ by identify the antipodal point, i.e. $p \sim -p$.

So when we remove a disk on $\mathbb{RP}^2$, it will become the quotient space of sphere removing two opposed disk. For example, we can just think $N := \mathbb{RP}^2 \backslash$a disk as the quotient space of $M_1 := \{(x_1, x_2, x_3) \in \mathbb{S}^2, -\frac{1}{2} < x_3 < \frac{1}{2}\}$.

Note that the set $M_2 := \{(x_1, x_2, x_3) \in \mathbb{S}^2, -\frac{1}{2} < x_3 < \frac{1}{2}, x_1 \leq 0\}$ already covers N under quotient map. So we can view N as the quotient space by identify the point $(0, x_2, x_3) \sim (0, -x_2, -x_3)$ on M_2, this exact the construction of Möbius band. (The only left thing is to construct a homeomorphism between M_2 and $[0,1] \times \mathbb{R}$)

The following is a picture in the construction.

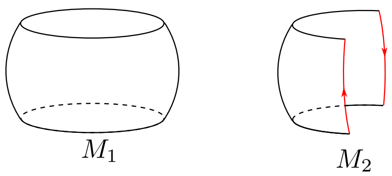

Figure 1: The pictures of M_1, M_2 and identification on boundary

Since M is non-orientable, $\mathbb{RP}^2$ is non-orientable, too since the orientation can be pass to the submanifold by restricting the atlas on submanifold.

Problem 2

We will use $[z_1, z_2]$ to denote the equivalent class in $\mathbb{CP}^1$.

Construct the map $f : \mathbb{S}^2 \to \mathbb{CP}^1$ by

$$f(x_1, x_2, x_3) := \begin{cases} [\frac{x_1+ix_2}{1-x_3}, 1], & x_3 \neq 1 \\ [1, \frac{x_1-ix_2}{1+x_3}], & x_3 \neq -1 \end{cases}$$

We need to verify f is well-defined when $x_3 \neq 1, -1$. Indeed, we have (Note $x_1 + ix_2 \neq 0$.)

$$\left[\frac{x_1 + ix_2}{1 - x_3}, 1\right] = \left[1, \frac{1 - x_3}{x_1 + ix_2}\right] = \left[1, \frac{(1 - x_3)(x_1 - ix_2)}{x_1^2 + x_2^2}\right] = \left[1, \frac{x_1 - ix_2}{1 + x_3}\right]$$

which shows f is well-defined.

Now let's show f is a diffeomorphism.

Let $(U_1, \phi_1), (U_2, \phi_2)$ be the two charts on $\mathbb{S}^2$ defined as

$$U_1 = \mathbb{S}^2 \backslash \{(0, 0, 1)\}, \phi_1(x_1, x_2, x_3) = (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3})$$
$$U_2 = \mathbb{S}^2 \backslash \{(0, 0, -1)\}, \phi_2(x_1, x_2, x_3) = (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3})$$

Let $(V_1, \varphi_1), (V_2, \varphi_2)$ be the two charts on $\mathbb{CP}^1$ defined by

$$V_1 = \mathbb{CP}^1 \backslash \{[1, 0]\}, \varphi_1([z_1, z_2]) = \frac{z_1}{z_2}$$
$$V_2 = \mathbb{CP}^1 \backslash \{[0, 1]\}, \varphi_2([z_1, z_2]) = \frac{z_2}{z_1}$$

So for $p \in U_1$, f has the form under the chart (U_1, ϕ_1) and (V_1, φ_1) as following

$$\varphi_1 \circ f \circ \phi_1^{-1}(u_1, u_2) = u_1 + iu_2$$

which is a smooth function.

For $p \in U_2$, we have

$$\varphi_2 \circ f \circ \phi_2(u_1, u_2) = u_1 - iu_2$$

which is also smooth.

Hence f is a diffeomorphism.

Problem 3

Recall the $SO(n)$ is defined as following

$$SO(n) = \{B \in \mathbb{R}^{n \times n} : B^T B = I_n \text{ and } \det(B) = 1\}$$

So for any fixed $A \in SO(n)$, we know near A, we can write M as $M = f^{-1}(0)$ with $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, f(B) = B^T B$. We can drop the condition of $\det(B) = 1$ since in the sufficient small neighborhood $U \subset \mathbb{R}^{n \times n}$ of A with $\det(B) > 0$ for all $B \in U$. The condition $B^T B = I_n$ will force $\det(B) = 1$.

Hence we have $T_A M = \ker(df_A)$. We need to calculate $df_A : T_A(\mathbb{R}^{n \times n}) \to T_{I_n}(\mathbb{R}^{n \times n})$. Actually we can identify $T_A(\mathbb{R}^{n \times n})$ with $\mathbb{R}^{n \times n}$ for short notation. Hence for any $P \in \mathbb{R}^{n \times n}$, we have

$$df_A(P) = \lim_{t \to 0} \frac{f(A + tP) - f(A)}{t} = A^T P + P^T A$$

Note df_A is a surjective to the symmetric metrics, so dim $\ker(df_A) = n^2 - \frac{(n+1)n}{2} = \frac{n(n-1)}{2}$.

Hence $SO(n)$ has dimension $\frac{n(n-1)}{2}$ and the tangent space of $SO(n)$ at A is the space $\{P \in \mathbb{R}^{n\times n} : A^T P + P^T A = 0\}$.

Problem 4

Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in I}$ be an atlas of M^m. Then we let

$$\tilde{A} := \{(TU_i, \tilde{\phi}_i) : i \in I\} \text{ with } \tilde{\phi}_i(p, v) = (\phi(p), d\phi_p(v)) \in \phi(U_i) \times \mathbb{R}^m$$

The transition maps between $(TU_i, \tilde{\phi}_i), (TU_j, \tilde{\phi}_j)$ is

$$\Phi_{ij}(x, w) = (\phi_j \circ \phi_i^{-1}(x), d(\phi_j \circ \phi_i^{-1})_x(w))$$

Note that $d(\phi_j \circ \phi_i^{-1})_x$ is linear, so the Jacobian matrix is just itself. Hence

$$d\Phi_{ij}(x, w) = \begin{bmatrix} d\left(\phi_j \circ \phi_i^{-1}(x)\right) & 0 \\ 0 & d\left(\phi_j \circ \phi_i^{-1}(x)\right) \end{bmatrix}$$

Hence $\det(d(\Phi_{ij})) = \left[d(\phi_j \circ \phi_i^{-1}(x))\right]^2 > 0$ since $d(\phi_j \circ \phi_i^{-1}(x))$ non-degenerate.

This means all the transition maps are orientation-preserving. Hence TM is orientable.

Problem 5

(a) First, let's suppose $\pi : E \to B$ is trivial. Then there is a diffeomorphism $h : E \to B \times \mathbb{R}^n$ with $h|_{\pi^{-1}(x)}$ is an isomorphism when restriction on the fiber $\pi^{-1}(x)$ for $x \in B$.

Let $\{e_1, \cdots, e_n\}$ be the canonical basis in $\mathbb{R}^n$ and we choose n maps $\{s_i\}_{1\le i\le n}$ by $s_i(b) = h^{-1}(b, e_i)$. Now we will show each s_i will be a sections.

Clearly $s_i : B \to E$ is smooth since h is a diffeomorphism. Note that since $h|_{\pi^{-1}(b)}$ is an isomorphism between $\pi^{-1}(b)$ and $\{b\} \times \mathbb{R}^n$, which means $h^{-1}|_{\{b\}\times\mathbb{R}^n}(b, e_i) \in \pi^{-1}(b)$ and hence $\pi \circ s_i(b) = b$, which shows s_i is indeed a section.

Moreover, we know that $\{s_i(b)\}_{1\le i\le n}$ forms a linearly independent set of $\pi^{-1}(b)$ since h is an isomorphism on $\pi^{-1}(b)$.

Secondly, let's assume there is n linearly independent sections $\{s_i\}_{1\le i\le n}$. Let's define the map $h : E \to B \times \mathbb{R}^n$ by the following method.

For each $p \in B$, let $b = \pi(p)$. Since $\{s_i(b)\}_{1\le i\le n}$ forms a linearly independent set of $\pi^{-1}(b)$, we can find unique $(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ with $p = \sum_{i=1}^n a_i s_i(b)$. Then we define $h(p) := (\pi(p), (a_1, a_2, \cdots, a_n)) \in B \times \mathbb{R}^n$. Note that the inverse of h is also well-defined and has form $h^{-1}(b, (a_1, \cdots, a_n)) = \sum_{i=1}^n a_i s_i(b)$ as the linear space. So we know that $h|_{\pi^{-1}(p)} : \pi^{-1}(b) \to \{b\} \times \mathbb{R}^n$ is an isomorphism.

Now let's verify h is a diffeomorphism. Let (U_i, ϕ_i) be a local trivialization of E near $\pi(p)$. I.e. $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ is a diffeomorphism with $\pi(p) \in B$ and the restriction on each fiber is an isomorphism. Now since s_i is a (smooth) section of $\pi : E \to B$, which means $\phi_i \circ s_i : U_i \to U_i \times \mathbb{R}^n$ smooth. So the map $\phi_i \circ h^{-1} : U_i \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$ is smooth with respect to the first variable. But $\phi_i \circ h^{-1}$ is linear(isomorphism) with respect to the second variable, $\phi_i \circ h^{-1}$ is indeed smooth. Moreover, we can write $\phi_i \circ h^{-1}(b, (a_1, \cdots, a_n)) = (b, \sum_{j=1}^n a_j \pi_2 \circ \phi_i \circ s_j(b))$ where π_2 is the projection from $U_i \times \mathbb{R}^n \to \mathbb{R}^n$. So we can find the differential of $\phi_i \circ h^{-1}$ is always non-degenerate. This shows both h and h^{-1} are locally diffeomorphism and hence h is indeed a diffeomorphism.

(b) Let B be the quotient space $[0, 1]$ where we identify the 0 and 1. We can easily find B is diffeomorphic to S^1. Let $V_1 = (0, 1) \subset B$, $V_2 =$ the quotient of $(0.5, 1] \cup [0, 0.5)$. Let $\varphi_1([x]) = x$, $\varphi_2([x]) = x$ for $x > 0.5$ and $\varphi_2([x]) = x + 1$ for $x < 0.5$. So B can be covered by two charts $(V_1, \varphi_1), (V_2, \varphi_2)$.

With the notations in Problem 1, we can actually see that $(U_1, \phi_1), (U_2, \phi_2)$ give us a way to locally trivialize the space M over the base space B. Moreover, the transition map restricted on each fiber is an isomorphism. So the above will give the structure of vector bundle of $\pi : M \to S^1$ where $\pi([x, t]) = [x]$.

Now let's show $\pi : M \to S^1$ is non-trivial. If on the contrary, $\pi : M \to S^1$ is a trivial vector bundle, then by above, we can find a section $s : S^1 \to M$ such that $s(b) \neq 0$ on the fiber $\pi^{-1}(b)$. (This is because M is a rank 1 vector bundle.)

So in the local chart (V_1, φ_1) and local trivialization (U_1, ϕ_1), s can be written as $\phi_1 \circ s \circ \varphi_1^{-1}(x) = (x, s_1(x))$, where $s_1(x)$ decided by s which is non-zero everywhere for $x \in (0, 1)$. WOLG, we assume $s_1(x) > 0$ for $x \in (0, 1)$. If we work on the chart (V_2, φ_2) and local trivialization (U_2, ϕ_2), we can also get $\phi_2 \circ s \circ \varphi_2^{-1}(x) = (x, s_2(x))$ for some $s_2 : (0.5, 1.5) \to \mathbb{R}$ which nonzero everywhere and hence it does not change sign.

Recall the transition map $\phi_2 \circ \phi_1^{-1}(x, t)$, we know for $x > 0.5$, $s_2(x) = s_1(x)$ and $s_2(x) > 0$ for $0.5 < x < 1$. For $x < 0.5$, we have $s_2(x + 1) = -s_1(x)$ and $s_2(x) < 0$ for $1 < x < 1.5$. This is a contradiction with the above fact that $s_2(x)$ does not change sign.

Hence $\pi : M \to S^1$ is a non-trivial vector bundle.