Exercise 5

1. We summarize the properties of the Lebesgue measure on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$, let

$$\mathcal{L}^n(E) = \inf \left\{ \sum_k |C_k| : \ E \subset \bigcup_k C_k, C_k \text{ closed cubes} \right\} .$$

We have

(a) $\mathcal{L}^n(C) = 1$ for every unit cube C (open or closed).

(b) $\mathcal{L}^n$ is a σ-finite Borel measure.

(c) $\mathcal{L}^n$ is finite on bounded sets.

(d) For every measurable E,

$$\mathcal{L}^n(E) = \inf \left\{ \mathcal{L}^n(G) : \ E \subset G, \ G \text{ open} \right\} ;$$

$$\mathcal{L}^n(E) = \sup \left\{ \mathcal{L}^n(K) : \ K \subset E, \ K \text{ compact} \right\} .$$

(e) Let T be a linear transformation from $\mathbb{R}^n$ to itself. For each measurable E, $T(E)$ is also measurable and there is some constant C_T such that

$$\mathcal{L}^n(T(E)) = C_T \mathcal{L}^n(E) .$$

(a)-(d) were covered in previous exercises. Prove (e).

2. Let Φ be a Lipschitz continuous map on $\mathbb{R}^n$ to $\mathbb{R}^n$, that is, for some $L > 0$,

$$|\Phi(x) - \Phi(y)| \leq L|x - y| , \quad \forall x, y \in \mathbb{R}^n .$$

Show that $\Phi(E)$ is measurable if E is (Lebesgue) measurable.

3. This problem is related to the σ-finiteness condition in Proposition 2.10. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, \ldots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy.$$

Let μ be the measure associated with this Λ by the representation theorem. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

4. Let μ be a Borel measure on $\mathbb{R}^n$ such that $\mu(K) < \infty$ for all compact K. Show that μ is the restriction of some Riesz measure on $\mathcal{B}$.