# TA's solution[1] to 5011 midterm exam

**Q1(a)** $S : X \to \overline{\mathbb{R}}$ defined by $S(x) := \sum_{k=1}^{\infty} \chi_{A_k}(x)$ is a measurable function, whence $E = S^{-1}(\{2020\})$ is measurable. Alternatively, since

$$E = \bigcup_{i_1 < \cdots < i_{2020}} \left( A_{i_1} \cap \cdots \cap A_{i_{2020}} \cap \bigcap_{k \notin \{i_1, \ldots, i_{2020}\}} A_k^c \right)$$

and $\{(i_1, \ldots, i_{2020}) \in \mathbb{N}^{2020} : i_1 < \cdots < i_{2020}\}$ is a subset of the countable set $\mathbb{N}^{2020}$, we see that $E$ is measurable.

**(b)** Please refer to assignment 1 solution for a proof.

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**Q2(a)** We should proceed with caution for not having $\infty - \infty$. Consider the following example. If we write

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right],$$

then

(a) when $f \equiv \infty$ and $g \equiv -\infty$, we have $fg \equiv -\infty$ while $(f+g)$ is undefined;

(b) when $f \equiv \infty$ and $g \equiv \infty$, we have $fg \equiv \infty$ while $(f-g)$ is undefined;

(c) when $f \equiv \infty$ and $g \equiv 0$, we have $fg \equiv 0$ while $[(f+g)^2 - (f-g)^2]$ is undefined.

To proceed, we may assume $f, g$ are real-valued functions (rather than extended real-valued) when answering this question. We then refer to lecture notes Ch1 Proposition 1.3 for a proof.

**(b)** Let $\mathcal{F} := \{E \in \mathcal{P}_{\mathbb{R}} : f^{-1}(E) \in \mathcal{B}\}$. We first show that $\mathcal{F}$ is a $\sigma$-algebra:

- Since $f^{-1}(\mathbb{R}) = \mathbb{R} \in \mathcal{B}$, we have $\mathbb{R} \in \mathcal{F}$;
- If $E \in \mathcal{F}$, then $f^{-1}(E) \in \mathcal{B}$, whence $f^{-1}(\mathbb{R} \setminus E) = \mathbb{R} \setminus f^{-1}(E) \in \mathcal{B}$, which shows $\mathbb{R} \setminus E \in \mathcal{F}$;
- If $E_i \in \mathcal{F}$, then $f^{-1}(E_i) \in \mathcal{B}$ for all $i$, whence $f^{-1}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{B}$, which shows $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$.

Since $f$ is continuous, $\mathcal{F}$ contains all open sets in $\mathbb{R}$. As $\mathcal{B}$ is the smallest $\sigma$-algebra containing all open sets in $\mathbb{R}$, we have $\mathcal{B} \subseteq \mathcal{F}$. Consequently, for all $B \in \mathcal{B}$, we have $B \in \mathcal{F}$, whence $f^{-1}(B) \in \mathcal{B}$.

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**Q3(a)** Plainly $\mu([a,b]) \le \phi([a,b])$. To show the reverse inequality, let $\{I_k = [a_k, b_k]\}_{k=1}^{\infty}$ be a collection of closed and bounded intervals such that $[a,b] \subseteq \bigcup_k I_k$. Our aim is to show

$$\sum_{k=1}^{\infty} \phi(I_k) \ge \phi([a,b]) = g(b) - g(a).$$

[1] This solution is adapted from the work by former TAs.

*Approach 1*[2]

Recall that $g$ is a continuous, non-decreasing function on $\mathbb{R}$. By $[a,b] \subseteq \bigcup_k [a_k, b_k]$, we claim that $[g(a), g(b)] \subseteq \bigcup_k [g(a_k), g(b_k)]$, which may be justified as follows. Given $y \in [g(a), g(b)]$, by the intermediate value theorem, there exists $x \in [a,b]$ such that $y = g(x) \in g([a,b]) \subseteq g(\bigcup_k [a_k, b_k]) \subseteq \bigcup_k [g(a_k), g(b_k)]$.

As a result,

$$g(b) - g(a) = \mathcal{L}([g(a), g(b)]) \leq \mathcal{L}(\bigcup_{k=1}^{\infty} [g(a_k), g(b_k)]) \leq \sum_{k=1}^{\infty} \mathcal{L}([g(a_k), g(b_k)]) = \sum_{k=1}^{\infty} \phi(I_k),$$

which was to be demonstrated.

*Approach 2*[3]

Fix an $\varepsilon > 0$. Since $g$ is continuous and non-decreasing, there exist $r_k, s_k$ such that

$$\begin{cases} -\infty < r_k < a_k \leq b_k < s_k < \infty \\ g(s_k) - g(b_k) < \varepsilon/2^{k+1} \\ g(a_k) - g(r_k) < \varepsilon/2^{k+1}. \end{cases}$$

It follows that we have

$$[a,b] \subseteq \bigcup_k I_k \subseteq \bigcup_k (r_k, s_k) \subseteq \bigcup_k [r_k, s_k],$$

and

$$\varepsilon + \sum_{k=1}^{\infty} \phi(I_k) \geq \sum_{k=1}^{\infty} \phi([r_k, s_k]).$$

As $[a,b]$ is compact and covered by $\{(r_k, s_k)\}$, there is a finite sub-covering, say, $\{(r_k, s_k)\}_{k=1}^N$. Let $\{C_\ell\}_{\ell \in L}$ be the connected components of the set $\bigcup_{k=1}^N [r_k, s_k]$. Since $[a,b]$ a connected subset of $\bigcup_{k=1}^N [r_k, s_k]$, it is contained in, say, $C_1$. Given $1 \leq k \leq N$, as $[r_k, s_k]$ is connected, we have either $[r_k, s_k] \subseteq C_1$ or $[r_k, s_k] \cap C_1 = \emptyset$. Therefore, $C_1 = \bigcup_{k \in K} [r_k, s_k]$, where $K := \{1 \leq k \leq N : [r_k, s_k] \subseteq C_1\}$. Since connected subsets of $\mathbb{R}$ are exactly singletons and intervals, we see that $C_1$ is a closed interval, which we denote by $[E_{\min}, E_{\max}]$.

Let $E := \{r_k\}_{k \in K} \cup \{s_k\}_{k \in K}$ be the set of all end points given by $[r_k, s_k], k \in K$. Given $e \in E$ with $e \neq E_{\max}$, we use $e^\uparrow$ to denote the immediate successor of $e$ in $E$. i.e. $e^\uparrow$ is the smallest element in $E$ which is greater than $e$. Noting that $E \subseteq [E_{\min}, E_{\max}]$, we have $[e, e^\uparrow] \subseteq [E_{\min}, E_{\max}] = \bigcup_{k \in K} [r_k, s_k]$, whence there exists $k \in K$ such that

$$\frac{e + e^\uparrow}{2} \in [r_k, s_k].$$

As a result,

$$\begin{cases} e < \dfrac{e + e^\uparrow}{2} \leq s_k \Rightarrow e^\uparrow \leq s_k \\ e^\uparrow > \dfrac{e + e^\uparrow}{2} \geq r_k \Rightarrow e \geq r_k. \end{cases}$$

[2] A student suggests this idea.
[3] A student suggests this idea.

i.e. $[e, e^{\uparrow}] \subseteq [r_k, s_k]$. Consequently, each such $[e, e^{\uparrow}]$ is contained in some $[r_k, s_k]$, whence

$$\begin{aligned}\varepsilon + \sum_{k=1}^{\infty} \phi(I_k) &\geq \sum_{k=1}^{\infty} \phi([r_k, s_k]) \geq \sum_{k\in K} \phi([r_k, s_k]) = \sum_{k\in K} \sum_{\substack{e\in E\setminus E_{\max} \\ [e,e^{\uparrow}]\subseteq[r_k,s_k]}} \phi([e, e^{\uparrow}]) \\ &= \sum_{e\in E\setminus E_{\max}} \phi([e, e^{\uparrow}]) \sum_{\substack{k\in K \\ [e,e^{\uparrow}]\subseteq[r_k,s_k]}} 1 \geq \sum_{e\in E\setminus E_{\max}} \phi([e, e^{\uparrow}]) \\ &= \phi([E_{\min}, E_{\max}]) \geq \phi([a, b]) \quad \text{since } [a, b] \subseteq [E_{\min}, E_{\max}].\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the result follows.

**(b)** Let $\mathcal{G}$ be the collection of all closed and bounded intervals in $\mathbb{R}$. As $(\mathcal{G}, \phi)$ forms a gauge, $\mu$ is an outer measure on $\mathbb{R}$. We shall apply Caratheodory's criterion to show that $\mu$ is a Borel measure. So pick two sets $E, F \subseteq \mathbb{R}$ with $\delta_1 := \mathrm{dist}(E, F) > 0$. We want to show that $\mu(E\cup F) = \mu(E)+\mu(F)$. By subadditivity of $\mu$ we only need to show that $\mu(E \cup F) \geq \mu(E) + \mu(F)$.

Let $\varepsilon > 0$. By cutting intervals into smaller ones, we see that

$$\mu(E) = \inf \left\{ \sum_k \phi(I_k) : E \subseteq \bigcup_k I_k, I_k \text{ closed and bounded interval with } \mathrm{diam}(I_k) < \delta_1/2 \right\}.$$

Therefore, we can find a countable collection $\mathcal{I}$ of closed intervals such that $E \cup F \subseteq \bigcup_{J\in\mathcal{I}} J$,

$$\mu(E \cup F) + \varepsilon \geq \sum_{J\in\mathcal{I}} \phi(J),$$

and $\mathrm{diam}(J) < \delta_1/2$ for all $J \in \mathcal{I}$. Thus each $J \in \mathcal{I}$ can only intersect at most one of $E$ and $F$. Let $\mathcal{I}_1 := \{J \in \mathcal{I} : J \cap E \neq \emptyset\}$ and $\mathcal{I}_2 := \{J \in \mathcal{I} : J \cap F \neq \emptyset\}$. We have $E \subseteq \bigcup_{J\in\mathcal{I}_1} J$, $F \subseteq \bigcup_{J\in\mathcal{I}_2} J$, and $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, whence

$$\begin{aligned}\mu(E \cup F) + \varepsilon &\geq \sum_{J\in\mathcal{I}} \phi(J) \\ &\geq \sum_{J\in\mathcal{I}_1} \phi(J) + \sum_{J\in\mathcal{I}_2} \phi(J) \\ &\geq \mu(E) + \mu(F).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, The result follows.

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**Q4(a)** The answer is no. To construct a counter example, let $h : [0, 1] \to [0, 2]$ be the function given by lecture notes Ch3 section 3.2. i.e. $h(x) := x + g(x)$ where $g$ is the Cantor function. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x) := \begin{cases} x & \text{if } x < 0 \\ h(x) & \text{if } 0 \leq x \leq 1 \\ x + 1 & \text{if } 1 < x. \end{cases}$$

Using the property of $h$, we see that $\Phi$ is an injective and continuous function on $\mathbb{R}$. Denoting the Cantor set by $\mathcal{C}$, we have $\mathcal{L}(\Phi(\mathcal{C})) = \mathcal{L}(h(\mathcal{C})) = 1$ by the property of $h$. Therefore, by lecture notes Ch3 Proposition 3.3, there exists some non-measurable $A \subseteq \Phi(\mathcal{C})$. Since $\Phi$ is injective, $E := \Phi^{-1}(A)$ is a subset of $\mathcal{C}$. As $\mathcal{C}$ is of measure zero, $E$ is a measurable set, while $\Phi(E) = A$ is not measurable.

**(b)** Please refer to assignment 5 solution and remark 5 for detail.

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**Q5** In the following, the whole space in consideration is $[0,1]$, so that for $E \subseteq [0,1]$, $E^c = [0,1] \setminus E$.

*Approach 1*[4]

Let $\varepsilon > 0$. Define $F_n := \{x \in [0,1] : f(x) > 1/n\}$. Since $F_n \subseteq F_{n+1}$ and $\bigcup_n F_n = [f > 0]$, we have $1 = \mathcal{L}([f > 0]) = \lim_{n\to\infty} \mathcal{L}(F_n)$, whence there exists $N$ such that $\mathcal{L}(F_N) \geq 1 - \varepsilon$. It follows that

$$\begin{aligned}\mathcal{L}(E_k) &= \mathcal{L}(E_k \cap F_N) + \mathcal{L}(E_k \cap F_N^c) \\ &\leq N \int_{E_k \cap F_N} f d\mathcal{L} + \mathcal{L}(E_k \cap F_N^c) \\ &\leq N \int_{E_k} f d\mathcal{L} + \mathcal{L}([0,1]) - \mathcal{L}(F_N) \\ &\leq N \int_{E_k} f d\mathcal{L} + \varepsilon.\end{aligned}$$

Letting $k \to \infty$, we have $\overline{\lim}\mathcal{L}(E_k) \leq \varepsilon$. The result follows.

*Approach 2*

It is proved by contradiction. Suppose there exists $\varepsilon > 0$ such that $\mathcal{L}(E_k) \geq \varepsilon$ for infinitely many $k$. By considering subsequence, we may assume $\mathcal{L}(E_k) \geq \varepsilon$ for all $k$. As $\int \left| f\chi_{E_k} \right| d\mathcal{L} \to 0$, by lecture notes Ch1 Proposition 1.20, there exists subsequence $\{k_j\}$ and a null set $U \subseteq [0,1]$ such that $\lim_{j\to\infty} f(x)\chi_{E_{k_j}}(x) = 0$ for all $x \in U^c$.

Claim that $U^c \cap [f > 0] \subseteq \bigcup_{L=1}^{\infty} \bigcap_{j \geq L} E_{k_j}^c$. To justify this, note that if $x \in U^c \cap [f > 0]$, then $\lim_{j\to\infty} \chi_{E_{k_j}}(x) = 0$. Since $\chi_{E_{k_j}}(x)$ can only be zero or one, this means $\chi_{E_{k_j}}(x) = 0$ for all but a finite number of $j$, whence $x \in \bigcup_{L=1}^{\infty} \bigcap_{j \geq L} E_{k_j}^c$.

Consequently, $1 = \mathcal{L}(U^c \cap [f > 0]) \leq \lim_L \mathcal{L}(\bigcap_{j \geq L} E_{k_j}^c) \leq \overline{\lim}_L \mathcal{L}(E_{k_L}^c) \leq 1 - \varepsilon$, which is a contradiction.

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**Q6** Plainly $\mu$ is a nonnegative function on $\mathcal{M}$ and $\mu(\emptyset) = 0$. Let $\{E_k\}$ be a countable collection of mutually disjoint sets in $\mathcal{M}$. Writing $E := \bigcup_k E_k$, we would like to show that

$$\mu(E) = \sum_k \mu(E_k).$$

On the one hand, given $F_0 \in \mathcal{M}$, we have

$$\begin{aligned}\sum_k \mu(E_k) &= \sum_k \inf\{\mu_1(E_k \setminus F) + \mu_2(E_k \cap F) : F \in \mathcal{M}\} \\ &\leq \sum_k [\mu_1(E_k \setminus F_0) + \mu_2(E_k \cap F_0)] = \mu_1(E \setminus F_0) + \mu_2(E \cap F_0),\end{aligned}$$

[4] A student suggests this solution.

whence $\sum_k \mu(E_k) \leq \mu(E)$ by taking inf over $F_0 \in \mathcal{M}$ on the R.H.S.

To get the reverse inequality, let $\varepsilon > 0$ be fixed. For each $k$, there exists $F_k \in \mathcal{M}$ such that

$$\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k) \leq \mu(E_k) + \frac{\varepsilon}{2^k}$$

Let $F := \bigcup_k (E_k \cap F_k)$. Note that $F \subseteq E$ and $E \setminus F = \bigcup_k (E_k \setminus F_k)$. Hence

$$\begin{aligned}
\mu(E) &\leq \mu_1(E \setminus F) + \mu_2(E \cap F) \\
&= \sum_k \mu_1(E_k \setminus F_k) + \sum_k \mu_2(E_k \cap F_k) \\
&= \sum_k [\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k)] \\
&\leq \sum_k \mu(E_k) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof.