TA's remarks on 5011 homework 8

1. The mark distribution for Hw8 is:
   Q2(a), 2(d), 3(a), 3(d), 5 (2 marks each).

2. In the second part of Q2(b), suppose $[r, s] \subseteq E$ and we want to show that φ is continuous at s. Note that for $t \in (r, s)$, we have $|f(x)|^t \le \left| f(x)\chi_{[|f|>1]}(x) \right|^s + \left| f(x)\chi_{[|f|\le 1]}(x) \right|^r \le |f(x)|^s + |f(x)|^r$. Since the R.H.S. is integrable, it follows from the dominated convergence theorem that

$$\lim_{t\uparrow s} \varphi(t) = \lim_{t\uparrow s} \int_X |f|^t = \int_X \lim_{t\uparrow s} |f|^t = \int_X |f|^s = \varphi(s).$$

3. To answer Q2(d), note that from Q2(a) we have

$$\begin{aligned} \|f\|_p^p = \int_X |f|^p \le \left(\int_X |f|^r\right)^\lambda \left(\int_X |f|^s\right)^{1-\lambda} &= \|f\|_r^{r\lambda} \|f\|_s^{s(1-\lambda)} \\ &\le (\max\{\|f\|_r, \|f\|_s\})^{r\lambda} (\max\{\|f\|_r, \|f\|_s\})^{s(1-\lambda)} \\ &= \max\{\|f\|_r, \|f\|_s\}^p. \end{aligned}$$

   Observe that we cannot reach the answer if we consider "$\varphi(r) \le \max\{\varphi(r), \varphi(s)\}$" instead. The reason may be that $\varphi(r), \varphi(s)$ are "unnormalized" while $\|f\|_r, \|f\|_s$ are "normalized". For instance, consider the case where f is a constant function and $\mu(X) = 1$.

4. In the solution to Q3(d), the monotone convergence theorem is used. The argument may be as follows[1]. Since the map $p \mapsto |f(x)|^p$ is convex for $|f(x)| \in (0, \infty)$ (by checking second derivative), we have $\frac{|f(x)|^p - 1}{p}$ increases as $p > 0$ increases. Hence, given $|f(x)| \in (0, 1)$, we have $0 \le \frac{1 - |f(x)|^p}{p}$ increases as p decreases towards zero. Therefore, the monotone convergence theorem can be applied to give

$$\lim_{p\downarrow 0} \int_{[|f|<1]} \frac{1 - |f|^p}{p} = \int_{[|f|<1]} \lim_{p\downarrow 0} \frac{1 - |f|^p}{p} = \int_{[|f|<1]} -\log|f|.$$

5. As suggested by the NOTES AND COMMENTS section of Rudin's *Real and Complex Analysis*, a very complete answer to Q4 was found by A. Villani (1985). So by using citation, we can effortlessly submit a probably very complete answer to this question.

[1] A student suggests this argument.