TA's remarks on 5011 homework 7

1. In this 12-question warm-up exercise, you can gain 10/12 marks from each answered question. On the other hand, there is mark deduction if your Hw2-5 are overdue. The arrangement is as follows.

No. of late submittion	Mark deduction
1	0.5
2	1.5
3	3
4	5

For example, if you have answered 11 questions in Hw7, and you have submitted Hw1,3,5 late while you haven't submitted Hw2,4, then your marks for Hw7 is

$$11 \times \frac{10}{12} - 1.5 = 7.67.$$

2. The TA is not familiar with functional analysis, so he is not so qualified to review the homework. Sorry about that.

3. Nevertheless, let's try to fill in some of the blank of the solution. The first is Q5. Let S be a dense set of ℓ^∞. To show that S is uncountable, a formal way is to find an injective map from some uncountable set to S.

   Since S is dense in ℓ^∞, for each $a \in \{0,1\}^{\mathbb{N}}$, there exists $s_a \in S$ such that $\|s_a - a\|_\infty < 0.5$. Consider the function $f : \{0,1\}^{\mathbb{N}} \to S$ defined by $f(a) := s_a$. If $f(a) = f(b)$ for some $a, b \in \{0,1\}^{\mathbb{N}}$, then $\|a-b\|_\infty \leq \|a - f(a)\|_\infty + \|f(b) - b\|_\infty < 1$, whence $a = b$ and f is injective. This shows $\mathrm{card}(\{0,1\}^{\mathbb{N}}) \leq \mathrm{card}(S)$.

4. In Q9(c), to find a sequence $\{f_n\} \in X$ such that $\|g - f_n\| \to 1$, an approach is as follows. Define a continuous function $h_n : [0,1] \to \mathbb{R}$ by

$$h_n(x) := \begin{cases} 0 & \text{if } x = 0 \\ \text{linear} & \text{if } x \in (0, \delta_n) \\ 1 + \dfrac{1}{n} & \text{if } x \in [\delta_n, 1], \end{cases}$$

   where δ_n is a small number such that $\int_0^1 h_n \geq 1 + \frac{1}{2n}$. The function $\Phi : [0,1] \to \mathbb{R}$ defined by $\Phi(\theta) := \int_0^1 (g - \theta h_n)$ is continuous. As $\Phi(0) = 1 > 0$ and $\Phi(1) \leq 1 - (1 + \frac{1}{2n}) < 0$, by the intermediate value theorem there exists $\theta_0 \in [0,1]$ such that $\Phi(\theta_0) = 0$. If we define $f_n|_{[0,1]} := g|_{[0,1]} - \theta_0 h_n$, then $\int_0^1 f_n = 0$ and $\|g - f_n\|_{[0,1]} = \theta_0 \cdot \|h_n\|_{[0,1]} \leq 1 + \frac{1}{n}$. We can define $f_n|_{[-1,0]}$ similarly, so that the resulting f_n is continuous on $[-1,1]$ with $f_n(0) = g(0)$, $f_n \in X$, and $\|g - f_n\| \leq 1 + \frac{1}{n}$.