TA's remarks on 5011 homework 3

1. The mark distribution for Hw3 is: Q1 (2 marks); Q2 (3 marks); Q4 (3 marks); Q5 (2 marks).

2. In Q4, we should verify if $\widetilde{\mu}$ is well-defined. That is, given a set $E \subseteq X$, if there exist $A_1, A_2, B_1, B_2 \in \mathcal{M}$ such that $A_i \subseteq E \subseteq B_i$ and $\mu(B_i \setminus A_i) = 0$, then whether $\mu(A_1) = \mu(A_2)$. Also, we should check whether $\widetilde{\mu}(E) \in [0, \infty]$ for all $E \in \widetilde{\mathcal{M}}$.

3. Q4 and Q5 involves three σ-algebra ($\mathcal{M}$, $\widetilde{\mathcal{M}}$, $\mathcal{M}_1$) and two measures (μ, $\widetilde{\mu}$). Therefore, when discussing "null sets", "measure zero sets", etc, we better specify the associated σ-algebra/measure. E.g. "null sets with respect to μ", "measure zero sets in $\mathcal{M}$". Also, we note that $\mu(E)$ may be undefined for $E \in \widetilde{\mathcal{M}}$.

4. Q5 may be rephrased as follows. Let $\mathcal{N} := \{U \in \mathcal{P}_X : U \subseteq A \text{ for some } A \in \mathcal{M} \text{ with } \mu(A) = 0\ \}$. Denote by $\sigma(\mathcal{M}, \mathcal{N})$ the smallest σ-algebra on X containing every element of $\mathcal{M}$ and $\mathcal{N}$. That is,

   - $\sigma(\mathcal{M}, \mathcal{N})$ is a σ-algebra on X, and we have $\mathcal{M}, \mathcal{N} \subseteq \sigma(\mathcal{M}, \mathcal{N})$.
   - If Σ is a σ-algebra on X such that $\mathcal{M}, \mathcal{N} \subseteq \Sigma$, then $\sigma(\mathcal{M}, \mathcal{N}) \subseteq \Sigma$.

   (c.f. Lecture notes Chapter 1 p.2 for its existence) Q5 then asks for showing that $\sigma(\mathcal{M}, \mathcal{N}) = \widetilde{\mathcal{M}}$.

5. Rudin's *Real and Complex Analysis* chapter 1, section "The Role Played by Sets of Measure Zero" gives us a better understanding of Q4 and Q5. An excerpt:

   > [the exercise] says that every measure can be completed, so, whenever it is convenient, we may assume that any given measure is complete; this just gives us more measurable sets, hence more measurable functions.

6. Q6 concerns measure extensions, which for example can be used in probability theory. Chow & Teicher's *Probability Theory* suggests

   > A salient underpinning of probability theory is the one-to-one correspondence between distribution functions on $\mathbb{R}^n$ and probability measures on the Borel subsets of $\mathbb{R}^n$. Verification of this correspondence involves the notion of measure extension.

   The idea may be roughly described as follows. Suppose we are given a distribution functions F of a random variable X, and we want to know $\Pr(X \in B)$ for all Borel sets B. Then Pr may be constructed by extending the basic rule "$\Pr(X \in (a, b]) := F(b) - F(a)$ for all $a, b \in [-\infty, \infty]$".

   The construction of product space is another example of measure extensions. Given two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, we may try to construct $(X \times Y, \mathcal{M} \times \mathcal{N}\ (?), \mu \times \nu\ (?))$ by extending the basic rule "$(\mu \times \nu)(M \times N) := \mu(M) \cdot \nu(N)$ for all $M \in \mathcal{M}, N \in \mathcal{N}$".

7. Q8 asks

   > Let $\mathcal{R}$ be the collection of all closed cubes in $\mathbb{R}^n$. A closed cube is of the form $I \times \cdots \times I$ where I is a closed, bounded interval. Show that $(\mathcal{R}, |\cdot|)$ forms a gauge.

   The definition of a closed cube should be rewritten as

   > A closed cube is of the form $I_1 \times \cdots \times I_n$ where I_i is a closed, bounded interval with $|I_1| = \cdots = |I_n|$.

   The first definition actually works for $(\mathcal{R}, |\cdot|)$ to form a gauge. But then something interesting happens. E.g. Let $n = 2$ and the resulting outer measure be μ. Then for the singleton $E := \{(0, 1)\}$, we have $\mu(E) \geq 1$. It is because for $[a, b] \times [a, b] \ni (0, 1)$, we have $[a, b] \supseteq [0, 1]$.