Solution to MATH5011 homework 2

(1) Let g be a measurable function in $[0, \infty]$. Show that

$$m(E) = \int_E g \, d\mu$$

defines a measure on $\mathcal{M}$. Moreover,

$$\int_X f \, dm = \int_X fg \, d\mu, \qquad \forall f \text{ measurable in } [0, \infty].$$

Solution: We readily check that

(1) $m(\phi) = 0$;

(2) $m(E) \geq 0$, $\forall E \in M$;

(3) For mutually disjoint $A_k \in M$,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \int_X \sum_{k=1}^{\infty} \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} \int \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} m(A_k)$$

by monotone convergence theorem, since $\sum_{k=1}^{n} \chi_{A_k} g \uparrow \sum_{k=1}^{\infty} \chi_{A_k} g$.

To prove the last assertion, consider the following cases:

(a) $f = \chi_E$ for some $E \in M$.

$$\int_X f \, dm = \int_E \, dm = m(E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu = \int_X fg \, d\mu.$$

(b) f is a non-negative simple function.

This follows from (a).

(c) f is a non-negative measurable function.

Pick a sequence $s_n \geq 0$ of simple functions such that $s_n \uparrow f$ pointwisely.

Then $0 \le s_n g \uparrow g$ pointwisely. From (b),

$$\int_X s_n\, dm = \int_X s_n g\, d\mu.$$

Taking $n \to \infty$, by monotone convergence theorem, we have

$$\int_X f\, dm = \int_X fg\, d\mu.$$

(2) Let $\{f_k\}$ be measurable in $[0, \infty]$ and $f_k \downarrow f$ a.e., f measurable and $\displaystyle\int f_1\, d\mu < \infty$. Show that

$$\lim_{k\to\infty} \int f_k\, d\mu = \int f\, d\mu.$$

What happens if $\displaystyle\int f_1\, d\mu = \infty$?

Solution: From the assumption we know the integrability of f_1 implies that all f_k are integrable. Without loss of generality, we may suppose $f_k \downarrow f$ pointwisely. (Otherwise, replace by X by $Y = X \setminus N$, such that $\mu(N) = 0$ and $f_k \downarrow f$ on Y.) Then $0 \le f_1 - f_k \uparrow f_1 - f$. By monotone convergence theorem,

$$\lim_{k\to\infty} \int_X (f_1 - f_k)\, d\mu = \int_X (f_1 - f)\, d\mu.$$

Since $\displaystyle\int_X f_1\, d\mu < \infty$, we can cancel it from both sides to yield the result.

If $\int_X f_1\, d\mu = \infty$, the result does not hold. For example, one may take $X = \mathbb{R}$, $f_k(x) = 1/k$ and $f = 0$. Then

$$\int_X f\, d\mu = 0, \text{ while } \int_X f_k\, d\mu = \infty, \ \forall k \in N.$$

(3) Let f be a measurable function. Show that there exists a sequence of simple functions $\{s_j\}$, $|s_1| \le |s_2| \le |s_3| \le \cdots$, and $s_k(x) \to f(x)$, $\forall x \in X$.

Solution: Choose sequences of non-negative simple functions $s_j^+ \uparrow f_+$ and

$s_j^- \uparrow f_-$. Put $s_j = s_j^+ \chi_{\{x:f(x)\geq 0\}} - s_j^- \chi_{\{x:f(x)<0\}}$. Fix $x \in X$. If $f(x) \geq 0$ then $|s_j(x)| = s_j^+(x) \uparrow f_+$. If $f(x) < 0$ then $|s_j(x)| = s_j^-(x) \uparrow f_-$. We also have

$$s_j(x) \to f_+ \chi_{\{x:f(x)\geq 0\}}(x) - f_- \chi_{\{x:f(x)<0\}}(x) = f(x), \quad \forall x \in X.$$

(4) Let $\mu(X) < \infty$ and f be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in [a,b], \ \forall E \in \mathcal{M}, \mu(E) > 0$$

for some $[a,b]$. Show that $f(x) \in [a,b]$ a.e..

Solution: Let $A = \{x : f(x) < a\}$ and $B = \{x : f(x) > b\}$. If $\mu(A) > 0$, we will draw a contradiction. Let $A_n = \{x \in A : f(x) < a - 1/n\}$ so $A = \bigcup_n A_n$. As $\{A_n\}$ is an ascending family tending to A, we can find some n_0 such that $\mu(A_{n_0}) > 0$. Then

$$\frac{1}{\mu(A_{n_0})} \int_{A_{n_0}} f d\mu \leq a - \frac{1}{n_0} \ ,$$

contradiction. Similarly we can treat the case $\mu(B) > 0$.

(5) Let f be Lebsegue integrable on $[a,b]$ which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every c. Show that f is equal to 0 a.e..

Solution: We can express our assumption as

$$\int_a^c f_+ d\mathcal{L}^1 = \int_a^c f_- d\mathcal{L}^1, \qquad \forall c \in [a,b] \ .$$

Clearly this implies these two integrals holds when (a,c) is replaced by any open interval. As every open set in $[a,b]$ can be written as a countable disjoint union of open intervals, these two integrals are equal over any open set. From Lebesgue integration theory we know that for every Lebesgue measurable E, there is an open set G containing E with the approximating measure. Thus we conclude

$$\int_E f_+ d\mathcal{L}^1 = \int_E f_+ d\mathcal{L}^1,$$

for all measurable E. Taking $E = \{x \in [a,b] : \ f(x) > 0$, we see that $\int_E f_+ d\mathcal{L}^1 = 0$, which implies $f \leq 0$ a.e.. By taking $E = \{x : \ f(x) < 0\}$, we see that $f \geq 0$ a.e. . Hence $f = 0$ a.e. .

(6) Let $f \geq 0$ be integrable and $\displaystyle\int f \, d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n\to\infty} \int n \log\left(1 + \left(\frac{f}{n}\right)^\alpha\right) \, d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0,1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

Solution: Let $g_n(x) = n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right)$. Since $\displaystyle\int f \, d\mu = c \in (0, \infty)$, we know that $\mu(\{x : f(x) = \infty\}) = 0$ and $\mu(\{x : f(x) > 0\}) > 0$. Observe that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \infty, & \text{on } \{x : f(x) > 0\}, \text{ if } \alpha < 1, \\ f(x), & \text{a.e. } \mu, \text{ if } \alpha = 1, \\ 0, & \text{a.e. } \mu, \text{ if } \alpha > 1. \end{cases}$$

Moreover, if $\alpha \geq 1$, using the elementary inequalities $1 + x^\alpha \leq (1+x)^\alpha$ and $\log(1+x) \leq x$ for $x \geq 0$, we have

$$g_n \leq n \log\left(1 + \frac{f}{n}\right)^\alpha \leq n\alpha \cdot \frac{f}{n} = \alpha f \in L^1(\mu).$$

- Suppose $\alpha \in (0,1)$. By Fatou's lemma,

$$\varliminf_{n\to\infty} \int g_n \, d\mu \geq \int \varliminf_{n\to\infty} g_n \, d\mu = \infty.$$

Hence, $\displaystyle\lim_{n\to\infty} \int g_n \, d\mu = \infty$.

- Suppose $\alpha = 1$. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int g_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu = \int f \, d\mu = c.$$

- Suppose $1 < \alpha < \infty$. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int g_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu = 0.$$

(7) Let $\mu(X) < \infty$ and $f_k \to f$ uniformly on X and each f_k is bounded. Prove that

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Can $\mu(X) < \infty$ be removed?

Solution: We assume that $\mu(X) > 0$. (Otherwise, the result is trivial.) Let $\varepsilon > 0$ be given. Since $f_k \to f$ uniformly on X, there exists natural number N such that for all $k \geq N$ and for all $x \in X$, we have

$$|f_k(x) - f(x)| < \frac{\varepsilon}{\mu(X)}.$$

So, for all $k \geq N$, we have

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| \leq \int |f_k - f| \, d\mu < \varepsilon.$$

The result follows.

If $\mu(X) = \infty$, the result no longer holds. One may take $X = R$, $f_k(x) = 1/k$, $f(x) = 0$ and μ to be the Lebesgue measure. Then $f_k \to f$ uniformly on X and each f_k is bounded,

$$\int f \, d\mu = 0, \text{ while } \int f_k \, d\mu = \infty, \ \forall k.$$

(8) Give another proof of Borel-Cantelli lemma Problem 7 in Ex 1 by integration theory. (Hint: Study $g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x)$.)

Solution: Let $\{A_k\}$ be measurable, $A = \{x \in X : x \in A_k \text{ for infinitely many } k\}$ and suppose $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Write

$$g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x).$$

Then $x \in A$ if and only if $g(x) = \infty$. By Fatou's lemma,

$$\int g \, d\mu \le \sum_{j=1}^{\infty} \int \chi_{A_j} \, d\mu = \sum_{j=1}^{\infty} \mu(A_j) < \infty.$$

As a consequence of Markov's inequality, g is finite a.e., the conclusion follows.

This problem shows the power by expressing things in terms of measurable functions.

(9) Let f be a Riemann integrable function on $[a, b]$ and extend it to $\mathbb{R}$ by setting it zero outside $[a, b]$.

(a) Show that f is Lebsegue measurable.

(b) Show that the Riemann integral of f is equal to $\int_{\mathbb{R}} f d\mathcal{L}^1$.

(c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on $[a, b]$ and converges pointwisely to some function which is not Riemann integrable.

Solution:

(a) We assume the result and notation in question 10 of exercise 1, by the proof of 10b), f is Riemann integrable on $[a, b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$. Then for all natural number n, we may find partition of [a,b], $P_n = \{a = z_0 < z_1 < ... < z_{m_n} = b\}$ such that

$$0 \leq \overline{R}(P_n, f) - \underline{R}(P_n, f) \leq \frac{1}{n},$$

define two sequence of step function in the following way, for all x in $[z_j, z_{j+1})$,

$$\varphi_n(x) = \inf\left\{f(x) : x \in [z_j, z_{j+1}]\right\} ,$$

and

$$\psi_n(x) = \sup\left\{f(x) : x \in [z_j, z_{j+1}]\right\}.$$

For all x in $[a, b]$

$$h(x) = \sup\left\{\varphi_n(x) : n \in N\right\}$$

and

$$g(x) = \inf\left\{\psi_n(x) : n \in N\right\} ,$$

h and g are obviously Lebesgue measurable, we also have $\varphi_n(x) \leq h \leq f \leq g \leq \psi_n(x)$. For any natural number n,

$$0 \leq \int_a^b (g - h) d\mathcal{L}^1 \leq \int_a^b (\psi_n - \varphi_n) d\mathcal{L}^1 = \overline{R}(P_n, f) - \underline{R}(P_n, f) \leq \frac{1}{n} ,$$

so we have $h = f = g$ a.e. and f is Lebesgue measurable.

(b) By taking refinement with the partition $\{a = z_0 < z_1 = a + (b-a)/n < .. < z_j = a + j(b-a)/n < .. < z_{m_n} = b\}$ if necessary, we may assume the norm of partition P_n in (a) tend to 0 as $n \to 0$. As φ_n and ψ_n are integrable and $|f(x)| \leq |\varphi_n(x)| + |\psi_n(x)|$ for all x in $[a, b]$, f is Lebesgue integrable and

$$\underline{R}(P_n, f) = \int_a^b \varphi_n d\mathcal{L}^1 \leq \int_a^b f d\mathcal{L}^1 \leq \int_a^b \psi_n d\mathcal{L}^1 = \overline{R}(P_n, f) .$$

Using result in 10(b) of Ex.1 and let n go to ∞, we have Riemann integral $= \int_R f d\mathcal{L}^1$.

(c) We consider the famous Dirichlet function g which is not Riemann integrable, $g(x)$ =1 if x is rational and $\in [0,1]$, $g(x)$ =0 otherwise. Let $\{q_n : n \in N\}$ be an enumeration of all rational number in $[0,1]$ and define

$$f_n = \sum_{i=1}^{n} \chi_{q_i} \ .$$

Then each f_n is obviously uniformly bounded Riemann integrable with zero integral and yet $\{f_n\}$ converges pointwisely to the Dirichlet function for all x in $[0,1]$.

(10) Let f be integrable in $(X, \mathcal{M}, \mu)$. Show that for each $\varepsilon > 0$, there is some δ such that

$$\int_E |f| < \varepsilon, \qquad \text{whenever } E \in \mathcal{M}, \ \mu(E) < \delta \ .$$

This is called the absolute continuity of an integrable function.

Solution. Assume on the contrary there is some $\varepsilon_0 > 0$ and $E_j, \mu(E_n) \leq 2^{-n}$, such that $\int_{E_n} |f| d\mu \geq \varepsilon_0$. Let $A_n = \bigcup_{j \geq n} E_j$. Then

$$\mu(A_n) \leq \sum_{j \geq n} \mu(E_j) \leq \sum_{j \geq n} \frac{1}{2^j} = \frac{1}{2^{n-1}} \ .$$

Let $A = \cap_n A_n$. As $\{A_n\}$ is descending and $\mu(A_1)$ is finite,

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0 \ ,$$

that is, A is of measure zero. On the other hand, we have $|f|\chi_{A_n} \leq |f|$, by the dominated convergence theorem we have

$$\int_A |f| d\mu = \lim_{n \to \infty} \int_{A_n} |f| d\mu \geq \varepsilon_0 > 0 \ ,$$

contradiction holds.