The Kakeya Conjecture

presentation at the New Wave Mathematics Lecture series

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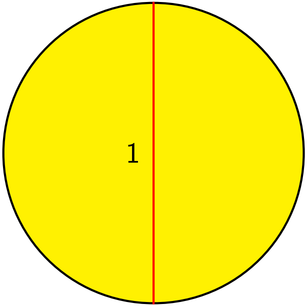
Kakeya's question (1917)

Soichi Kakeya
(1886-1947)

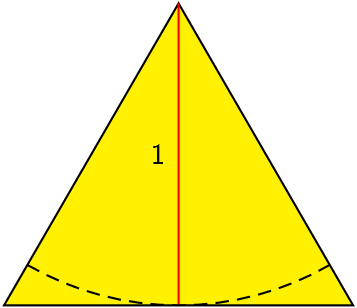
Suppose a needle of unit length can be turned through 180 degrees in a region in the plane, by rotations and translations only. What is the least area for such a region?

An obvious thought



$$\text{Area} = \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \simeq 0.785$$

A smaller area



$$\text{Area} = \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)(1) = \frac{1}{\sqrt{3}} \simeq 0.577$$

(If base length is x, then $x^2 = \left(\frac{x}{2}\right)^2 + 1^2$, which implies $x = \frac{2}{\sqrt{3}}$.)

Can the area be smaller still?

Abram Samoilovitch Besicovitch
(1891-1970)

Yes!

Besicovitch's construction (1928)

Indeed, the area can be made arbitrarily small!

Given any tiny positive number ε (say the the diameter of an atom), one can find a region D_ε in the plane, that

- D_ε has area smaller than that of ε; and yet
- a needle of unit length can be turned through 60 degrees inside D_ε, by translations and rotations only.

Splitting a triangle

Moving a sub-triangle

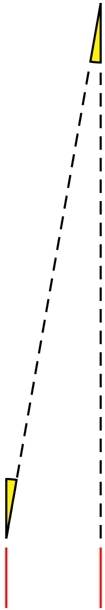
Moving a needle through 30 degrees

Jump! Then move through another 30 degrees

Jumping using Pál's joins

Gyula Pál (1881-1946)

We can move a unit line segment
to a parallel position
in an arbitarily small area!



Recap

So far we have found a region in the plane, with area quite a bit smaller than 0.577, in which a needle of unit length can be turned through 60 degrees.

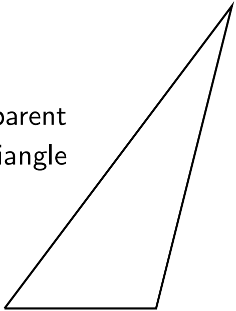
But this is not as small as possible yet.

To make this area smaller still, we will be carrying out the following construction a lot.

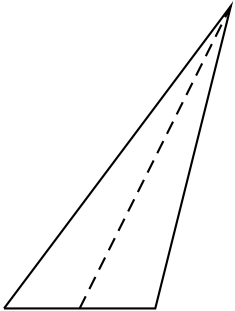
So let's introduce some terminologies.

Hearts and Arms

parent
triangle



Hearts and Arms



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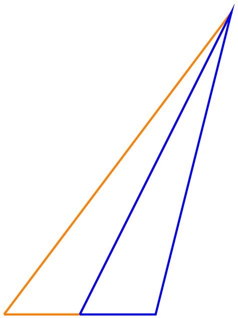

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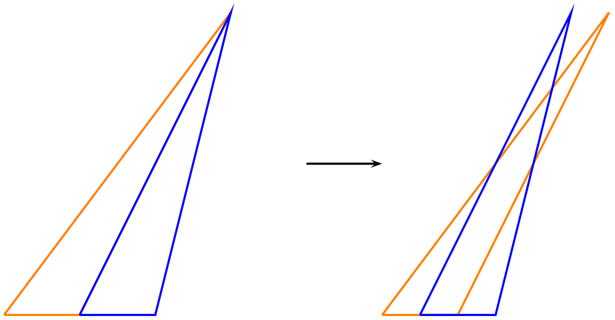
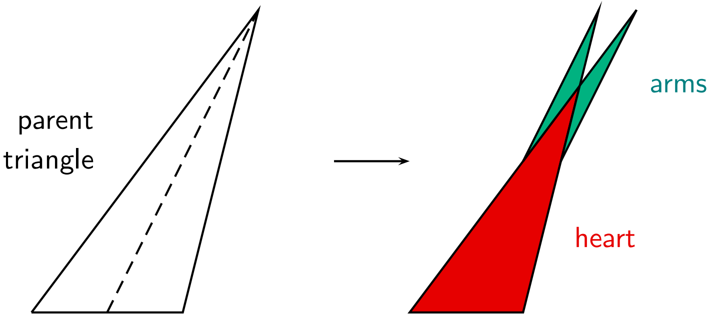


Hearts and Arms



Begin again... Split into 4 triangles!

Forming 4 level 1 arms, and 2 level 1 hearts

2 level 1 hearts recombine into one single heart

Repeat!

Forming 2 level 2 arms and 1 level 2 heart

Moving a needle through 60 degrees...
with 3 jumps!

Better still... Split into 8 triangles!

Forming 8 level 1 arms, and 4 level 1 hearts

4 level 1 hearts recombine into one single heart

Repeat:

Forming 4 level 2 arms and 2 level 2 hearts

2 level 2 hearts recombine into one single heart

Repeat:
forming 2 level 3 arms and 1 level 3 heart

In general, split the original triangle into 2^n pieces for some large positive integer n.

Pair the adjacent triangles to create 2^{n-1} pairs.

Create a heart and two arms within each pair. Then we get 2^{n-1} hearts, and 2^{n-1} pairs of arms.

The 2^{n-1} hearts form a triangle which is similar to the original one.

Then we iterate the above process, and re-translate the 2^n small triangles we begin with along the way.

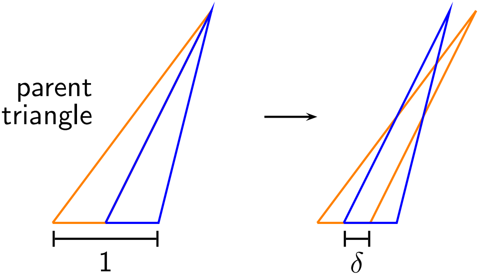
After n iterations, we then obtain a figure, which contains a unit line segment in each direction of a sector of 60 degrees, and whose area becomes very small if n is very large.

(Try this construction out with a program written by Terence Tao: http://www.math.ucla.edu/~tao/java/Besicovitch.html)

A lemma about areas

If $\dfrac{\text{distance moved}}{\text{original base length}} = \delta$, then

$$\text{Area of heart} = (1-\delta)^2(\text{Area of parent triangle})$$
$$\text{Total area of arms} = 2\delta^2(\text{Area of parent triangle})$$



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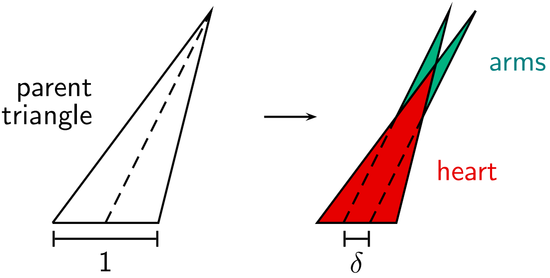


If we split a triangle of area 1 into 2^n pieces, and carry out the n steps mentioned before (keeping the same ratio δ at each step), then the area of the resulting figure after n steps is at most

$$\begin{aligned}
&\text{Total area of arms in all } n \text{ steps} + \text{Area of heart at the last step}\\
&= \left[2\delta^2 + 2\delta^2(1-\delta)^2 + \cdots + 2\delta^2(1-\delta)^{2(n-1)}\right] + (1-\delta)^{2n}\\
&\le \frac{2\delta^2}{1-(1-\delta)^2} + (1-\delta)^{2n}\\
&= \frac{2\delta}{2-\delta} + (1-\delta)^{2n},
\end{aligned}$$

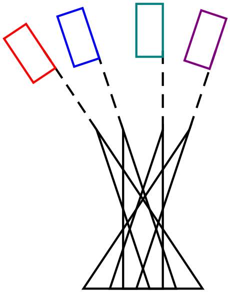
which can be made as small as one wishes, by first taking δ to be very small, and then n to be very large.

Application to wave equations

The above construction yields 2^n overlapping triangles, each with area $\simeq 2^{-n}$.

They all point in slightly different directions.

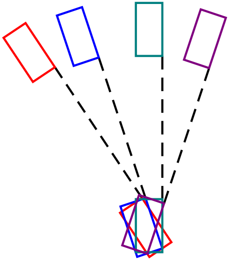
One can place 2^n rectangles, each with area $\simeq 2^{-n}$, along the extensions of these triangles, and these 2^n rectangles will have no overlap at all.



If we move these non-overlapping rectangles back along the dotted lines, we get 2^n rectangles that overlap a lot.

Hence if we create a wave of amplitude 1 in each of these non-overlapping rectangles, and send them in the direction of the dotted lines, then the waves concentrate, after a short time, in a very small area with a very large amplitude.

This way we get some interesting solutions of the wave equation!



A moment of reflection

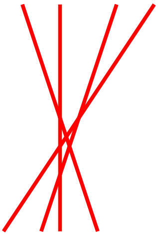
If a figure contains a unit line segment in every possible direction, we call the figure a *Kakeya set*.

We just saw that a Kakeya set in the plane can have an arbitrarily small area.

But this seems counter-intuitive: if a figure in the plane contains a unit line segment in every possible direction, how can it be very small?

Indeed, if we see it from the correct point of view, a Kakeya set cannot be too small.

One way of seeing this is to "thicken" the figure: we consider all points that are at distance at most h from the figure (think of h as very small, but positive).



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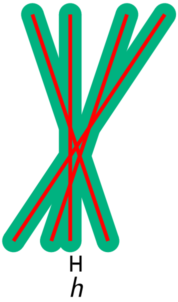


Theorem (Roy O. Davies 1971)

If E is a Kakeya set in the 2-dimensional plane, then after thickening it by h,

the area of the thickened figure is at least $A|\log h|^{-1}$

for some constant A independent of h.

The analog in dimensions 3 or above is wide open!

Kakeya conjecture in the Euclidean space

One way of formulating what might be true in 3-dimensions or above is the following:

Conjecture (Kakeya conjecture)

For any $\varepsilon > 0$, there exists a constant A_ε, such that the following is true:

If E is a Kakeya set in the n-dimensional space, $n \geq 3$, then after thickening it by h,

the volume of the thickened figure is at least $A_\varepsilon h^{-\varepsilon}$.

No one knows how to prove or disprove this yet, but there is an analog in algebra, that mathematicians know how to solve!

Detour: Modulo arithmetic

Suppose an integer is divided by some positive integer m. The remainder must be either $0, 1, 2, \ldots, m-1$.

We define addition of these m remainders by

$$a \oplus b = \text{the remainder of } a + b \text{ when divided by } m$$

and multiplication of these m remainders by

$$a \otimes b = \text{the remainder of } a \times b \text{ when divided by } m$$

e.g. $m = 2$

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

e.g. $m = 4$

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

e.g. $m = 5$

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

These “modulo arithmetic” obeys the usual rules for additions and multiplications. e.g.

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

$$a \oplus b = b \oplus a$$

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

$$a \otimes b = b \otimes a$$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

One can also solve algebraic equations like

$$a \otimes x = b$$

e.g. $m = 5$. Solve

$$3 \otimes x = 2.$$

Solution: We use the multiplication table for $m = 5$:

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The only solution to $3 \otimes x = 2$ is $x = 4$.

e.g. $m = 4$. Solve

$$2 \otimes x = 0.$$

Solution: We use the multiplication table for $m = 4$:

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Hence

$$2 \otimes x = 0 \quad \text{if and only if} \quad x = 0 \text{ or } x = 2.$$

So the equation $2 \otimes x = 0$ has 2 different solutions!

The problem here is that m is composite. If $m = ab$ is a product of two numbers a, b with $1 < a, b < m$, then

$$a \otimes b = 0,$$

as well as

$$a \otimes 0 = 0,$$

so we have two different solutions to the equation $a \otimes x = 0$.

This is not going to happen if $m = p$ is a prime: if $m = p$ is a prime, and a is one of the non-zero remainders $1, 2, \ldots, p - 1$, then the equation $a \otimes x = 0$ has only one solution, namely $x = 0$.

In fact, suppose $m = p$ is a prime, and a is one of the non-zero remainders $1, 2, \ldots, p-1$. Then x solves

$$a \otimes x = 0,$$

if and only if

$$ax \text{ is a multiple of } p.$$

which (since p is prime and $1 \leq a < p$) is equivalent to saying that

$$x \text{ is a multiple of } p,$$

i.e.

$$x = 0$$

if x is one of the numbers in $0, 1, 2, \ldots, p-1$.

Finite fields

Hence $\{0, 1, \ldots, p-1\}$, together with the addition $\oplus$ and the multiplication $\otimes$, is an example of what's called a *finite field* if p is a prime. In this case we write

$$\mathbb{F}_p = \{0, 1, \ldots, p-1\}.$$

With a little more work, one can show that if p is a prime, then for any a, b in $\mathbb{F}_p$ with $a \neq 0$, the equation

$$a \otimes x = b$$

has one and only one solution for x in $\mathbb{F}_p$.

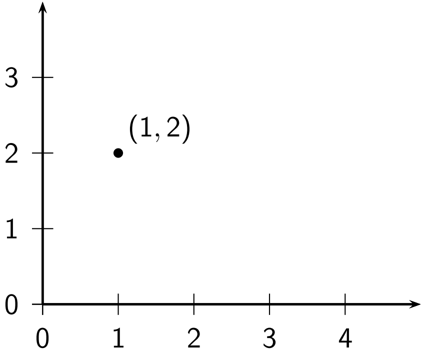
(Reason: We have already see the case when $b = 0$. Suppose now $b \neq 0$. Note that $a \otimes 1$, $a \otimes 2$, ..., $a \otimes (p-1)$ are non-zero, and they are all distinct. Hence they must be a reordering of $1, \ldots, p-1$, and exactly one of them must be b.)

Finite fields are very important objects in mathematics.

They are widely used in the study of number theory, and have real-world applications towards e.g. cryptography.

Another view at the plane

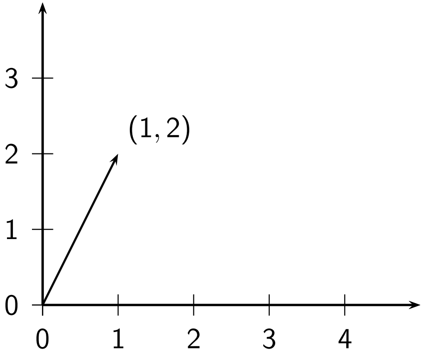
Given an ordered pair of real numbers (x, y), we can determine a point in the 2-dimensional plane:



(Hence the plane is sometimes written as $\mathbb{R}^2$, where $\mathbb{R}$ stands for the real numbers.)

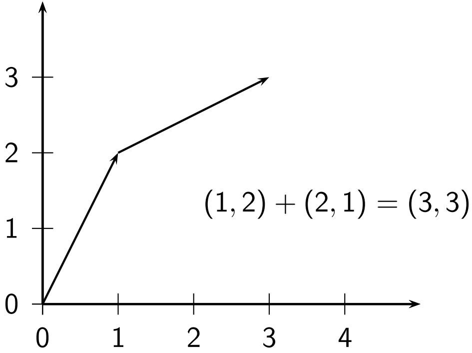
Given an ordered pair of real numbers (x, y), we can also determine a vector in the 2-dimensional plane:

$$x\mathbf{i} + y\mathbf{j} = (x, y)$$



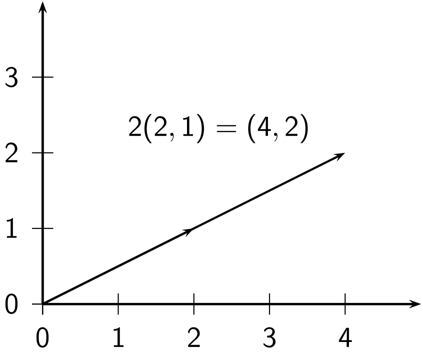
Vectors can be added:

$$(x, y) + (u, v) = (x + u, y + v)$$



Vectors can also be multiplied by a real number:

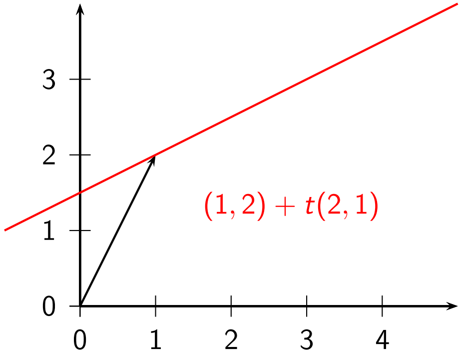
$$k(x, y) = (kx, ky)$$



Given a point (x, y) in the 2-dimensional plane, and a vector (u, v), the collection of points

$$(x, y) + t(u, v)$$

as t varies over the real numbers is a (straight) line that passes through (x, y), and points in the direction (u, v).



The same holds in higher dimensions: given n real numbers $x_1, x_2, \ldots, x_n$, we can identify the ordered tuple $(x_1, x_2, \ldots, x_n)$ with either a point in the n-dimensional space, or a vector in the same space.

A line through $(x_1, x_2, \ldots, x_n)$ pointing in the direction $(u_1, u_2, \ldots, u_n)$ can be described by

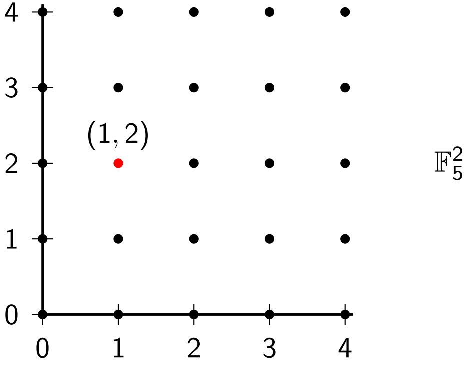
$$(x_1, x_2, \ldots, x_n) + t(u_1, u_2, \ldots, u_n)$$

where t varies over all real numbers.

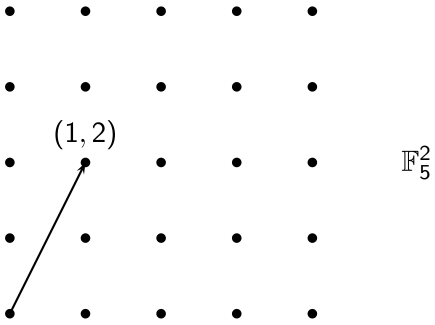
Idea: We can also replace the real numbers everywhere by numbers in $\mathbb{F}_p$ (where p is a prime).

A 2-dimensional plane over $\mathbb{F}_p$

Given an ordered pair (x, y), with x, y in $\mathbb{F}_p$, we can determine a point in the 2-dimensional plane $\mathbb{F}_p^2$:

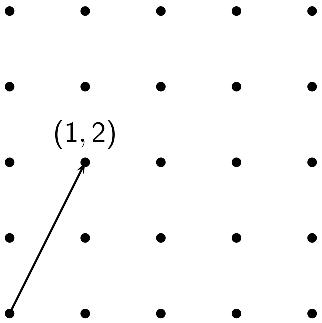


(so $\mathbb{F}_p^2$ contains exactly p^2 points) or a vector in $\mathbb{F}_p^2$:



Vectors in $\mathbb{F}_p^2$ can be added:

$$(x, y) \oplus (u, v) = (x \oplus u, y \oplus v)$$



$(1, 2) \oplus (2, 1) = ?$

in $\mathbb{F}_5^2$

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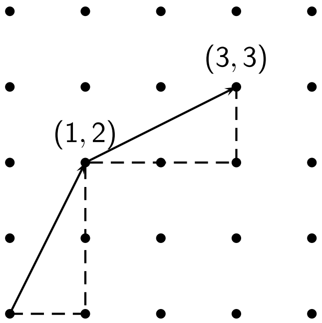

$$(1, 2) \oplus (2, 1) = (3, 3)$$

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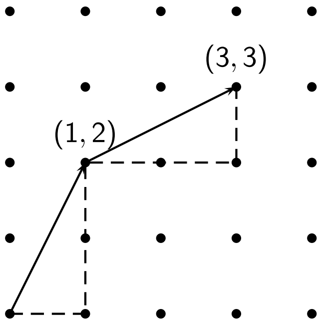

$(1, 2) \oplus (2, 1) = (3, 3)$

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(What about $(1, 2) \oplus (4, 2)$ in $\mathbb{F}_5^2$?)

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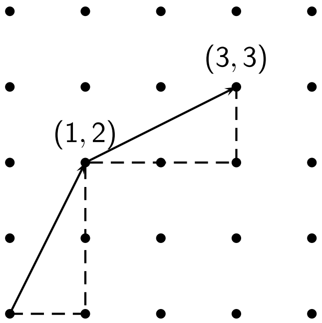


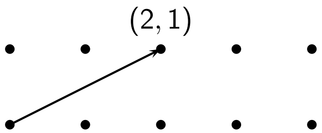
$(1, 2) \oplus (2, 1) = (3, 3)$

in $\mathbb{F}_5^2$

(What about $(1, 2) \oplus (4, 2)$ in $\mathbb{F}_5^2$? Answer: $(0, 4)$)

Vectors in $\mathbb{F}_p^2$ can also be multiplied by a number in $\mathbb{F}_p$:

$$k \otimes (x, y) = (k \otimes x, k \otimes y)$$



$2 \otimes (2, 1) = ?$

in $\mathbb{F}_5^2$

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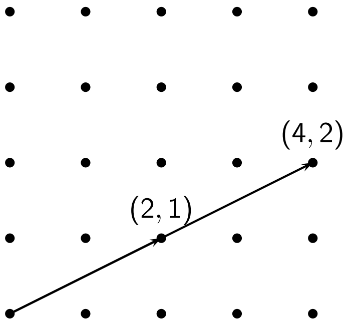

$2 \otimes (2, 1) = (4, 2)$ in $\mathbb{F}_5^2$

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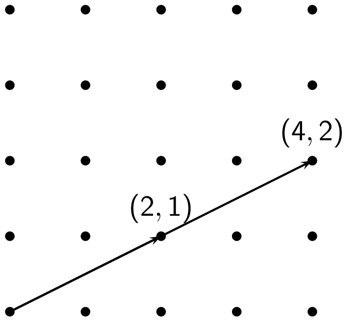

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(What about $3 \otimes (2, 1)$ in $\mathbb{F}_5^2$?)

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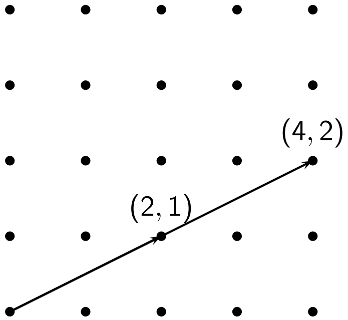

$2 \otimes (2, 1) = (4, 2)$

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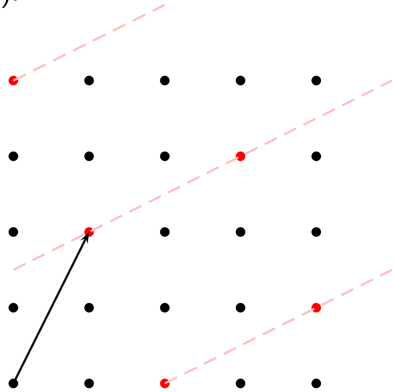
(What about $3 \otimes (2, 1)$ in $\mathbb{F}_5^2$? Answer: $(1, 3)$)

Given a point (x, y) in $\mathbb{F}_p^2$, and a vector (u, v) in $\mathbb{F}_p^2$, the collection of points

$$(x, y) \oplus [t \otimes (u, v)]$$

as t varies over $\mathbb{F}_p$ is a (straight) line that passes through (x, y), and points in the direction (u, v) (so now a line is just a collection of p points in $\mathbb{F}_p^2$).



$(1, 2) \oplus [t \otimes (2, 1)]$

in $\mathbb{F}_5^2$

The same holds in higher dimensions: given n numbers $x_1, x_2, \ldots, x_n$ in $\mathbb{F}_p$, we can identify the ordered tuple $(x_1, x_2, \ldots, x_n)$ with either a point in the n-dimensional space $\mathbb{F}_p^n$, or a vector in the same space.

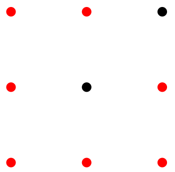
A line through $(x_1, x_2, \ldots, x_n)$ pointing in the direction $(u_1, u_2, \ldots, u_n)$ can be described by

$$(x_1, x_2, \ldots, x_n) \oplus [t \otimes (u_1, u_2, \ldots, u_n)]$$

where t varies over $\mathbb{F}_p$.

Kakeya sets over finite fields

A collection of points E in $\mathbb{F}_p^n$ is called a *Kakeya set*, if it contains a line in every possible direction.



An example of a Kakeya set in $\mathbb{F}_3^2$

Theorem (Dvir, 2008)

Let p be a prime, and n be a positive integer.

If E is a Kakeya set in $\mathbb{F}_p^n$, then E contains at least $A_n p^n$ points, where A_n is a constant that is independent of p.

This solves an analog of the Kakeya conjecture over finite fields.

The theorem actually works for all finite fields (not just those of the form $\mathbb{F}_p$).

It was first proved by Zeev Dvir in March 2008. The proof is only 5 pages long!

Main idea of the proof

Let's work over $\mathbb{F}_p$, where p is prime. For simplicity, if a, b are in $\mathbb{F}_p$, let's write $a \otimes b$ as ab, and $a \oplus b$ as $a + b$. If x is in $\mathbb{F}_p$, write x^n for $x \otimes x \otimes \cdots \otimes x$ (the product of n copies of x).

If $a_0, a_1, \ldots, a_d$ are in $\mathbb{F}_p$, we call

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

a one-variable polynomial of degree d, with coefficients in $\mathbb{F}_p$.

Lemma

If $a_0, a_1, \ldots, a_d$ are in $\mathbb{F}_p$, the polynomial equation

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 = 0$$

has at most d solutions in $\mathbb{F}_p$.

(Proof: Use the factor theorem.)

More generally:

Lemma

Suppose E is a collection of points in $\mathbb{F}_p$. Then

$$\textit{the number of points in } E \textit{ is at most } d,$$

if and only if

there exists a non-zero one-variable polynomial of degree $\leq d$, with coefficients in $\mathbb{F}_p$, that vanishes at every point in E.

Reason: $\Leftarrow$ was just the previous lemma. To see $\Rightarrow$, we need to find $a_0, a_1, \ldots, a_d$ in $\mathbb{F}_p$, not all zero, so that

$$\begin{cases} a_d x_1^d + a_{d-1} x_1^{d-1} + \cdots + a_0 & = 0 \\ a_d x_2^d + a_{d-1} x_2^{d-1} + \cdots + a_0 & = 0 \\ & \vdots \\ a_d x_k^d + a_{d-1} x_k^{d-1} + \cdots + a_0 & = 0 \end{cases}$$

where $x_1, x_2, \ldots, x_k$ is a listing of all points in E. If $k \leq d$, then there are more unknowns ($(d+1)$ of them) than equations (at most d of them), so we can find a non-zero solution as desired.

Continuing along these lines, it is not too difficult to prove the following two lemmas:

Lemma

Let E be a collection of points in $\mathbb{F}_p^n$, with less than C_n^{n+d} points. Then there is a polynomial P of n variables, with coefficients in $\mathbb{F}_p$ and of degree $\leq d$, which vanishes on E, but which is not identically zero. (Here C_r^n is the binomial coefficient $\frac{n!}{r!(n-r)!}$.)

Lemma

Let E be a collection of points in $\mathbb{F}_p^n$, which contains a line in every possible direction. Let P be a polynomial of n variables with coefficients in $\mathbb{F}_p$. If P vanishes on E, and the degree of P is $\leq p-1$, then P is identically zero.

Combining the above two lemmas, we immediately get the following corollary:

Corollary

Let E be a collection of points in $\mathbb{F}_p^n$, which contains a line in every possible direction. Then E contains at least C_n^{n+p-1} points.

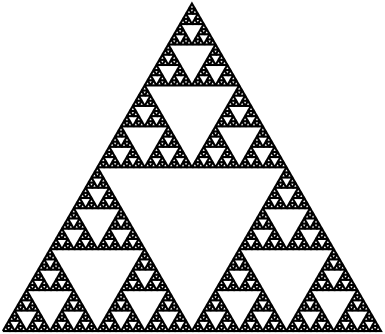
Since $C_n^{n+p-1} \geq A_n p^n$ as $p \to \infty$, this implies the theorem of Dvir we stated earlier.

Epilogue 1

There are a few more or less equivalent ways of formulating the Kakeya conjecture in $\mathbb{R}^n$.

Many of these involve a suitable notion of *dimension*, which we didn't have time to go into today.

(That'd be a very good excuse to talk about sets of fractional dimension, or *fractals*: can you imagine a set having dimension 1.585?)



The Sierpinski triangle

Epilogue 2

The study of Kakeya conjecture brings in tools from many areas of mathematics.

- Incidence geometry: e.g. work of Roe O. Davies, Antonio Cordoba, Stephen W. Drury, Michael Christ, Wilhelm Schlag, Thomas Wolff, Terence Tao, ...
- Additive combinatorics: e.g. work of Jean Bourgain, Nets Katz, Terence Tao, ...
- Heat flow: e.g. work of Jonathan Bennett, Anthony Carbery, Terence Tao, ...
- Algebraic topology: e.g. work of Larry Guth, Netz Katz, ...

Ideas related to the Kakeya conjecture has also found applications in many areas of mathematics:

- Fourier analysis: e.g. work of Charles Fefferman
- Wave equations: e.g. work of Thomas Wolff
- Analytic number theory: e.g. work of Jean Bourgain
- Cryptography: e.g. work of Jean Bourgain
- Random number generation: e.g. work of Zeev Dvir and Avi Wigderson
- Game theory: e.g. work of Yuval Peres (Microsoft)

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I also benefited from various articles on the blog of Terence Tao, which is an excellent source for those who would like some further reading.

http://terrytao.wordpress.com